Lecture-25

Stokes theorem :- Let M be a manifold, $\Omega \subseteq M$ a connected open subset, $\partial\Omega$ its topological boundary in M. Roughly, Stokes theorem states that

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega \quad \text{for } \omega \in \mathcal{D}^{m-1}(M);$$

We'll shortly give a more strict & precise statement.

— We'll call $\Omega \subseteq M$, Ω connected & open as a domain in M. Recall that the topological boundary of Ω is given by the set $\partial\Omega \subseteq M$ consisting of those points in M whose any open neighbourhoods intersect Ω as well as $M \setminus \Omega$.

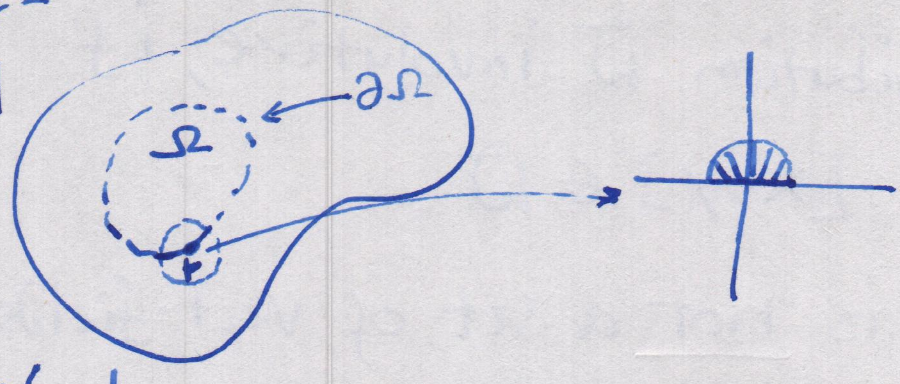
— We say Ω has smooth boundary if for every $p \in \partial\Omega$, $\exists$ a local chart (U, x) around p, with $x_i(p) = 0$, $1 \leq i \leq m = \dim M$, such that

$$U \cap \overline{\Omega} = \{ z \in U \mid x_m(z) \geq 0 \};$$

i.e. x_m maps $U \cap \overline{\Omega}$ into the upper half space $\mathbb{R}^m_+ := \{ y \in \mathbb{R}^m \mid y_m \geq 0 \}$.

25-2

M



We have

<u>Theorem</u>:- Let Ω be a domain in M, assume Ω has smooth boundary in M. Then $\partial\Omega$ is an $(m-1)$-dim'l closed submanifold of M. Moreover, if M is orientable, $\partial\Omega$ is orientable. $\boxtimes$

Starting with an oriented manifold M, we wish to fix an orientation on $\partial\Omega$, where Ω is a domain $\subseteq M$. First let $M = \mathbb{R}^{n+1}$ & $\Omega = \mathbb{R}^{n+1}_{>0} = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^{n+1}$ and $\{u_i\}$ its dual basis. Then

$$\overline{\Omega} = \{x \in \mathbb{R}^{n+1} \mid u_{n+1}(x) \geq 0\},$$

$$\partial\Omega = \{x \in \mathbb{R}^{n+1} \mid u_{n+1}(x) = 0\} \simeq \mathbb{R}^n.$$

On $\mathbb{R}^{n+1}$ the orientation is given by the $n+1$-volume form $dx_1 \wedge \cdots \wedge dx_{n+1}$.

25-3 We identify $T_x \mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ so that on each $T_x \mathbb{R}^{n+1}$ we have the form $u_1 \wedge \cdots \wedge u_{n+1}$ corresponding to $dx_1 \wedge \cdots \wedge dx_{n+1} \big|_x$.

The 'outward normal' to $\partial \Omega$ is given by the vector field $p \mapsto -\frac{\partial}{\partial x_{n+1}}\Big|_p$; with the usual identifications, this is $-e_{n+1}$.

On $T_p(\partial\Omega)$ we fix an orientation by requiring that $\{v_1, \dots, v_n\}$, a basis of $T_p(\partial \Omega)$, is positively oriented if

$$(u_1 \wedge \cdots \wedge u_{n+1})(-e_{n+1}, v_1, \dots, v_n) > 0.$$

(e.g. for $v_i := e_i$, we get

$$= \det \begin{pmatrix} -u_1(e_{n+1}) & u_1(e_1) & \cdots & u_1(e_n) \\ -u_2(e_{n+1}) & u_2(e_1) & \cdots & u_2(e_n) \\ \vdots & \vdots & & \vdots \\ -u_{n+1}(e_{n+1}) & u_{n+1}(e_1) & \cdots & u_{n+1}(e_n) \end{pmatrix}$$

$$= (-1)^{n+1}; \quad > 0 \iff n \text{ is odd}.$$

Thus a positive orientation for $\partial\Omega$ is given by the form $dx_1 \wedge \cdots \wedge dx_n$ if n odd & by $-dx_1 \wedge \cdots \wedge dx_n$, if n is even.

Now, for $\Omega \subseteq M$ a domain with smooth boundary, for $p \in \partial\Omega$, $\exists$ a +ve'ly oriented local chart $(U, x) \ni p$ in M s.t.

$$U \cap \partial\Omega = \{z \in U \mid x_m(z) = x_m(p)\}.$$

We require, on this chart, that the orientation is given by

Exc: The ordered basis $\{v_1, \dots, v_{m-1}\}$ of $T_p(\partial\Omega)$ is +ve'ly oriented if $\{N_p, v_1, \dots, v_{m-1}\}$ is a +ve'ly oriented, $N_p = -e_m$. (?)

$$\begin{cases} dy_1 \wedge \cdots \wedge dy_{m-1} & \text{if } m-1 \text{ is even (so } m \text{ odd)} \\ -dy_1 \wedge \cdots \wedge dy_{m-1} & \text{if } m-1 \text{ is odd (so } m \text{ even)} \end{cases}$$

If we fix this orientation on $\partial\Omega$, we say that $\partial\Omega$ has a <u>compatible orientation</u>.

We now can state

<u>Stokes' theorem</u>:- Let M be an oriented manifold, $\Omega \subseteq M$ a domain with smooth boundary, $\bar{\Omega}$ <u>compact</u>. Let $\omega \in D^{m-1}(M)$, $i : \partial\Omega \to M$ an embedding. Let $\partial\Omega$ have the compatible boundary orientation.

Then $$\int_\Omega d\omega = \int_{\partial\Omega} i^*(\omega).$$

$\boxtimes$

<u>Applications</u>: Stokes theorem generalizes the fundamental theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a). \text{ Here } \Omega = [a,b],$$

a 1-manifold with boundary $\partial\Omega = \{a,b\}$.

<u>Green's theorem</u>:- $D \subseteq \mathbb{R}^2$ a domain with smooth boundary, P, Q smooth functions on D. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} \underbrace{(P\,dx + Q\,dy)}_{\omega}$$

<u>Pf</u>: We have $\underbrace{d\left(P\,dx + Q\,dy\right)}_{d\omega} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$

Hence, by def^n of $\int$ of forms,

$$\int_{\partial D} \omega = \int_D d\omega = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$$

$$= \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Cor: Let M be a compact manifold without boundary ($\partial M = \phi$). Then for any $(m-1)$-form ω, $\int_M d\omega = 0$.

Cor:- Let M be a compact manifold with smooth boundary and ω a closed $(m-1)$-form on M (i.e. $d\omega = 0$). Then $\int_{\partial M} \omega = 0$.

Pf:- $\int_{\partial M} \omega = \int_M d\omega = 0$.

Exercise: Let $\omega = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ on S^1.

Then $\int_{S^1} \omega = 2\pi$.

By the above, we conclude that ω is not exact, however ω is closed (exc.).

25-7:

Lemma:- Let M be compact oriented. Then $\exists$ a diff. form ω of deg m s.t. every local representative form is positive everywhere on M.

Pf (Sketch): $\exists$ an oriented atlas with finitely many charts $\{V_i\}$. Let $\{\varphi_i\}$ be a partⁿ of unity subordinate to this. On each V_i, define $\omega_i = dx_1 \wedge \cdots \wedge dx_m$. Since the atlas is oriented, these remain +ve on overlaps of the V_i's.

Define $\omega = \sum \varphi_i \omega_i$. This is >0 & global. $\boxtimes$

Theorem:- There is no smooth retraction from a compact oriented manifold to its boundary.

Pf: (**Recall**: $r : M \to \partial M$ is a retraction if
$$r|_{\partial M} = \mathrm{Id}_{\partial M}).$$

Let ω be as in the Lemma. Since ∂M is $(m-1)$ dim'l, $d\omega = 0$ on ∂M. Suppose $r: M \to \partial M$ is a smooth retraction. We have $d(r^*\omega))$
$= r^*(d\omega) = 0$. Since $r|_{\partial M} = \mathrm{Id}_{\partial M}$, we have $r^*(\omega)|_{\partial M} = \omega$. Hence $\int_{\partial M} \omega = \int_{\partial M} r^*\omega$
$= \int_M d(r^*\omega) = \int_M r^*(d\omega) = 0 \rightarrow\leftarrow$ to $\int_{\partial M} \omega > 0$.