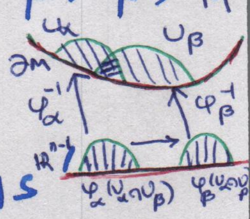
Lecture-26

Let us clarify a couple of ideas we discussed in the last class;

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n \geq 0\} \leftarrow \text{upper half space in } \mathbb{R}^n.$$

- A 'smooth manifold with boundary' is a T_2-Space M, having a countable basis of open sets & a 'differentiable structure' $\mathcal{A}$ in the following sense: $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is a family of open subsets U_α of M, with homeomorphisms φ_α onto an open subset of $\mathbb{R}^n_+$ (which has the subspace topology from $\mathbb{R}^n$) such that

i) $\bigcup_\alpha U_\alpha = M$ (ii) for $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathcal{A}$ with $U_\alpha \cap U_\beta \neq \varphi$, the maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ & $\varphi_\alpha \circ \varphi_\beta^{-1}$ are 'diffeomorphisms' of $\varphi_\alpha(U_\alpha \cap U_\beta)$ & $\varphi_\beta(U_\alpha \cap U_\beta)$, $\subseteq \mathbb{R}^n_+$.

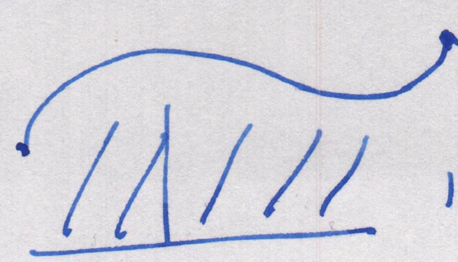


(iii) $\mathcal{A}$ is maximal satisfying (i) & (ii).

Examples: 1. Tori with 'discs' removed

2.

3.



$\mathbb{R}^n_+$.

<u>Remarks</u>: $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid x_n = 0\} \equiv \mathbb{R}^{n-1}$.

- If (U, φ) is a coordinate chart in M with boundary and if for $p \in U$, $\varphi(p) \in \partial \mathbb{R}^n_+$, then for <u>any</u> coordinate chart $(V, \psi) \ni p$, $\psi(p) \in \partial \mathbb{R}^n_+$ (Why?). We define the boundary of M by

$$\partial M = \{ p \in M \mid \varphi(p) \in \partial \mathbb{R}^n_+ \text{ for some chart } (U, \varphi) \ni p \}.$$

- Usual manifolds are regarded as manifolds with boundary with $\partial M = \phi$. In this case we also say M is a manifold without boundary. We mentioned last time that for M with $\partial M \neq \phi$, $\dim M = n$, ∂M carries a diff. structure determined by that on M and $\dim \partial M = n-1$; the inclusion $i : \partial M \to M$ is an embedding.
- When $\partial M = \phi$ & $\omega \in D^{n-1}(M)$, we <u>define</u>

$$\int_{\partial M} \omega = 0.$$

We'll now prove Stokes theorem $\rightarrow$

<u>Theorem</u> :- M be an oriented manifold, $\Omega \subseteq M$ a domain with smooth boundary, $\bar{\Omega}$ compact. Let $m = \dim M$, $\omega \in D^{m-1}(M)$, $i : \partial\Omega \to M$ an embedding and $\partial\Omega$ has the compatible orientation. Then

$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^*(\omega).$$

<u>Proof</u> :- We can choose a locally finite covering $\{U_\alpha\}$ of M such that

i) U_α is a chart $\forall \alpha$.

ii) If $U_\alpha \cap \bar{\Omega} \neq \phi$ then either $U_\alpha \subseteq \Omega$ or $U_\alpha \cap \partial\Omega \neq \phi$. If $U_\alpha \cap \partial\Omega \neq \phi$ we (may) assume $U_\alpha \subseteq \{p \mid |x_i^\alpha(p)| < \epsilon, 1 \leq i \leq m\}$; (recall $U_\alpha \cap \bar{\Omega} = \{q \mid x_m^\alpha(q) \geq 0\}$.), for any such U_α.

Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Since $\partial\Omega$ & $\bar{\Omega}$ are compact & $\{U_\alpha\}$ locally finite, the set $\{\alpha \mid U_\alpha \cap \partial\Omega \neq \phi$ or $U_\alpha \cap \bar{\Omega} \neq \phi\}$ is a <u>finite</u> set.

Hence $\displaystyle\int_\Omega d\omega = \sum_\alpha \int_\Omega d(f_\alpha \omega)$, $\omega = \sum_\alpha f_\alpha \omega$

and $\displaystyle\int_{\partial\Omega} i^*(\omega) = \sum_\alpha \int_{\partial\Omega} i^*(f_\alpha \omega)$.

→ Note that $\mathrm{Supp}(f_\alpha\omega) \subseteq U_\alpha$.

→ This reduces to proving: if M is a compact manifold (oriented) and $\partial\Omega$ has the compatible boundary orientation, then

$$\int_{\partial\Omega} i^*(\omega) = \int_\Omega d\omega,$$ where $\mathrm{Support}(\omega) \subseteq U_\alpha$ for some $U_\alpha =: U$.

→ If $U \cap \overline{\Omega} = \phi$ then ω and $d\omega$ both are zero on $\overline{\Omega}$ (Why?). Therefore both the integrals above are zero, so equal.

→ Assume therefore $U \cap \overline{\Omega} \neq \phi$. ~~If~~ First let us consider the case $U \subseteq \Omega$. Then $U \cap \partial\Omega = \phi$, hence $\displaystyle\int_{\partial\Omega} \omega = 0$.

Since ω is an $(n-1)$ from on M with $\mathrm{Supp}(\omega) \subseteq U$, We have $$\omega = \sum_{k=1}^{m} (-1)^{k-1} f_k \, dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_m$$

Hence $$d\omega = \sum_{k=1}^{m} (-1)^{k-1} df_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_m$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \left(\sum_{j=1}^{m} \frac{\partial f_k}{\partial x_j} dx_j \right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_m$$

$$= \sum_{k=1}^{m} \frac{\partial f_k}{\partial x_k} dx_1 \wedge \cdots \wedge dx_m .$$ Hence

$$\int_U d\omega = \sum_k \int_{\mathbb{R}^m} \frac{\partial f_k}{\partial x_k} dx_1 \cdots dx_m .$$

Now $$\int_{\mathbb{R}^m} \frac{\partial f_k}{\partial x_k} dx_1 \cdots dx_m = \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_k}{\partial x_k} dx_k \right) dx_1 \cdots \widehat{dx_k} \cdots dx_m .$$

Note that $\int_{-\infty}^{\infty} \frac{\partial f_k}{\partial x_k} dx_k$ is a function of the variables $x_1, \dots, \widehat{x_k}, \dots, x_m$; given by

$$(a_1, \dots, \widehat{a_k}, \dots, a_m) \longmapsto \int_{-\infty}^{\infty} g_k'(t)\, dt,$$ where

$$g_k(t) = f_k(x_1, \dots, \underset{k^{th}}{t}, \dots, x_m) .$$

Now f_k has compact support, hence g_k vanishes outside any suff. large interval $(-b, b) \subseteq \mathbb{R}$.

By the fundamental thm of calculus,

$$\int_{-\infty}^{\infty} g_k'(t)\,dt = \int_{-b}^{b} g_k'(t)\,dt = \underbrace{g_k(b)}_{=0} - \underbrace{g_k(-b)}_{=0} = 0.$$

It therefore follows that $\int_U d\omega = 0 = \int_\Omega d\omega$.

→ Now consider the case $U \not\subseteq \Omega$. Then ~~...~~ $U \cap \partial\Omega \neq \emptyset$. Observe that for $k \neq m$ the integration limits for g_k' are symmetric (since for $q \in U \cap \bar{\Omega}$, $x_m(q) \geq 0$), hence

$$\int_{-b}^{b} g_k'(t)\,dt = 0$$ for b large; by the above argument.

When $k = m$, on $\partial\Omega$ we have $x_m = 0$. In this case we have

$$i^*(\omega) = (-1)^{m-1} f_m(x_1, \ldots, x_{m-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{m-1}$$ on $\partial\Omega$.

Hence $$\int_U d\omega = \sum_{k=1}^{m} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} \frac{\partial f_k}{\partial x_k}\, dx_1 \wedge \cdots \wedge dx_m ,$$

Where ~~f_k~~ f_m vanishes if ~~x_k~~ x_m is outside some large interval $(0, \epsilon)$. Although $f_k(x_1, \ldots, \cancel{x_{k-1}}\, x_{m-1}, \epsilon) = 0$, $f_k(x_1, \ldots, \cancel{x_{k-1}}\, x_{m-1}, 0) \neq 0$ (possibly).

Again, applying the fundamental thm of Calculus we have

$$\int_U d\omega = \int_{\mathbb{R}^{m-1}} \int_0^{\epsilon} \frac{\partial f_m}{\partial x_m}\, dx_1 \wedge \cdot\, dx_{m-1} \wedge dx_m$$

$$= \int_{\mathbb{R}^{m-1}} f_m(x_1, \ldots, x_{m-1}, t)\Big|_0^{\epsilon} dx_1 \cdots dx_{m-1}$$

$$= -\int_{\mathbb{R}^{m-1}} f_m(x_1, \ldots, x_{m-1}, 0)\, dx_1 \cdots dx_{m-1}$$

On the other hand, on $\partial\Omega$, $i^*(\omega) = (-1)^{m-1} f_m(x_1, \ldots, x_{m-1}, 0)$.

→Now $\partial\Omega$ has orientation given by the form $(-1)^m dx_1 \wedge \cdots \wedge dx_{m-1}$. Therefore

$$\int_{\partial\Omega} i^*(\omega) = (-1)^m \int_{\mathbb{R}^{m-1}} (-1)^{m-1} f_m(x_1, \ldots, x_{m-1}, 0)\, dx_1 \cdots dx_{m-1}$$

$$= -\int f_m(x_1, \ldots, x_{m-1}, 0)\, dx_1 \cdots dx_{m-1}.$$

Hence we have $\int_\Omega d\omega = \int_{\partial\Omega} i^*(\omega)$. ∎