Lecture-3

In the last lecture we discussed path connectedness of some Lie groups, the case of $SL(n;\mathbb{R})$ was particularly interesting. A similar proof, with $A \in GL(n;\mathbb{C})$, $A^* = A$ define as positive if $\langle x, Ax\rangle > 0 \ \forall\, x \in \mathbb{C}^n - \{0\}$, where $\langle\, , \rangle$ is the (sesquilinear) inner product on $\mathbb{C}^n$; gives

<u>Theorem</u>:- (Polar decomposition for $SL(n;\mathbb{C})$): Given $A \in SL(n;\mathbb{C})$ $\exists$ a unique pair (U, P) with $U \in SU(n)$, $P = P^*$ and P positive, with $A = UP$; $\det P = 1$.

<u>Exercise</u>:- Polar decompositions for $GL(n;\mathbb{R})$, $GL(n;\mathbb{R})^+$ and $GL(n;\mathbb{C})$.

Let us now move towards the notion of Lie algebra for a Lie group G. Remember that G is a smooth manifold, with all points behaving alike, via the left / right translation maps, which are smooth.

Roughly speaking the Lie algebra of G consists of all smooth vector fields on G that are (left) translation invariant. Such fields are completely determined by their value at the identity element of G. Moreover, if X is such a vector field and $e \in G$ is the identity element, $a \in G$ any other point, to get $X(a)$, we use the translation by a and invariance of X. Given any element v of $T_e(G)$, the tangent space of G at e, One can construct a smooth vector field on G by translating v around on G. We need thus focus on $T_e(G)$ as far as left invariant vector fields on G are concerned. Recall the notion of an integral curve for a vector field. For a smooth manifold M and a smooth vector field X on M, an integral curve of X is a smooth curve $\alpha : (a,b) \to M$ such that the tangent vector to α at $t \in (a,b)$ is exactly the element $X(\alpha(t)) \in T_{\alpha(t)} M$, $\forall t \in (a,b)$. These curves are extremely usefull for calculus on M. For a matrix Lie group, the exponential map gives a uniform recipe for such curves.

<u>Exponential map</u>:- We identify $\underline{M_n(\mathbb{C})}$ with $\mathbb{C}^{n^2}$ and, for $X \in M_n(\mathbb{C})$, $\|X\| := \sqrt{\sum_{l,k=1}^{n} |x_{kl}|^2}$

Then $\|X+Y\| \leq \|X\| + \|Y\|$ and $\|XY\| \leq \|X\| \, \|Y\|$ $\forall X, Y \in M_n(\mathbb{C})$. We can talk of convergence of sequences of matrices as usual; $X_m \to X$ iff $\|X_m - X\| \to 0$. Cauchy sequences are defined.

It follows easily that $M_n(\mathbb{C})$ with this metric is complete, i.e. every Cauchy sequence converges.

An infinite series $\sum_{i=0}^{\infty} X_i$ is defined to converge absolutely if the series $\sum_{i=0}^{\infty} \|x_i\|$ converges.

If this is the case, then the sequence $(S_m)_m$ of partial sums $S_m = X_0 + \cdots + X_m$ is a Cauchy sequence, hence has a well defined limit, which we define as the sum $X_0 + X_1 + \cdots$

<u>Proposition 1</u>:- For any $X \in M_n(\mathbb{C})$, the series

$$e^X := \sum_{i=0}^{\infty} \frac{X^i}{i!}$$ converges. The map

$X \longmapsto e^X$ on $M_n(\mathbb{C})$ is continuous.

<u>Proof</u>:- We have $$\sum_{i=0}^{\infty} \left\| \frac{X^i}{i!} \right\| \leq \sum_{i=0}^{\infty} \frac{\|X\|^i}{i!} = e^{\|X\|}.$$

So the series e^X is absolutely convergent, hence is convergent.

Clearly each partial sum is a continuous func^n of X, the series of e^X converges uniformly on closed balls $\{\|Y\| \leq r\}$. Hence $X \mapsto e^X$ is continuous. $\boxtimes$

Some properties of e^X: Let $X, Y \in M_n(\mathbb{C})$.

(i) $e^0 = I$ (ii) $(e^X)^* = e^{X^*}$ (iii) $e^X \in GL_n(\mathbb{C})$,

$(e^X)^{-1} = e^{-X}$. (iv) $e^{(a+b)X} = e^{aX} e^{bX}$ $\forall$ $a, b \in \mathbb{C}$.

(v) If X, Y commute then $e^{X+Y} = e^X e^Y = e^Y e^X$.

(vi) For any $A \in GL_n(\mathbb{C})$, $Ae^X A^{-1} = e^{AXA^{-1}}$.

(vii) $\| e^X \| \leq e^{\|X\|}$.

Let us prove some of these: (i) is clear.

For (ii) Let $S_{m,*} := 1 + X^* + \frac{X^{*2}}{2!} + \cdots + \frac{X^{*m}}{m!}$.

Then $S_{m,*} = S_m^*$, where $S_m = 1 + X + \cdots + \frac{X^m}{m!}$.

Taking limits the result follows, as $X \mapsto X^*$ is continuous.

(v) Let $XY = YX$ and consider

$$e^X e^Y = \left(I + X + \frac{X^2}{2!} + \cdots\right)\left(I + Y + \frac{Y^2}{2!} + \cdots\right)$$

To compute the RHS, we may multiply the series term by term (?). Collecting total degree m terms we get

$$e^X e^Y = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{X^k}{k!} \frac{Y^{m-k}}{(m-k)!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k Y^{m-k}$$

Since $XY = YX$, we have the expansion

$$(X+Y)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k Y^{m-k}, \text{ so}$$

$$e^X e^Y = \sum_{m=0}^{\infty} \frac{1}{m!}(X+Y)^m = e^{X+Y}.$$

(vi) Follows since $(AXA^{-1})^i = AX^iA^{-1}$.

(vii) Left to you! $\boxtimes$

Coming back to the integral curves for vector fields, using the matrix exponential, we have

Theorem-3: Let $X \in M_n(\mathbb{C})$. Then $t \longmapsto e^{tX}$ defines a smooth curve in $M_n(\mathbb{C})$. Moreover,

$$\frac{d}{dt} e^{tX} = Xe^{tX} = e^{tX} \cdot X. \text{ Hence,}$$

$$\frac{d}{dt} e^{tX} \Big|_{t=0} = X.$$

Proof: We can differentiate the power series e^{tX} term by term, since for $1 \leq i,j \leq n$, $(e^{tX})_{ij}$ is a convergent power series in t etc. $\boxtimes$

Our next job is to get expression for e^X. We first do this for X diagonalizable, then nilpotent and then for general X using what is known as the Jordan decomposition for X

$X = S + N$, S diagonalizable, N nilpotent, $SN = NS$; this decomposition is unique such.

(we will discuss this later).

1. <u>Diagonalizable</u>:- Let $A \in GL(n;\mathbb{C})$ such that $AXA^{-1} = D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$e^{AXA^{-1}} = Ae^{X}A^{-1} = e^{D} = \mathrm{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}). \text{ So}$$

$$e^{X} = A^{-1}\,\mathrm{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})A.$$

2. <u>Nilpotent</u>:- In this case $X^m = 0$ for some $m > 0$. But then e^X is a polynomial in X.

3. <u>General case</u>:- Write $X = S + N$, S diagonalizable, N nilpotent and $NS = SN$. Then it follows that

$$e^{X} = e^{S+N} = e^{S}e^{N}$$

We have the recipe for e^S as well as e^N, so we have found an expression for e^X.

In the next class we will see some more properties of the matrix exponential and proceed towards the notion of Lie algebra of a Lie group.