Lecture-3

Calculus contd :- The class C^r, C^∞ etc: We saw that the existence and continuity of the partial derivatives of a function f on an open $U \subseteq \mathbb{R}^n$ implies differentiability of f at all points of U; If $f : U \to \mathbb{R}^m$ is differentiable on U, we have $Df : U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$, $a \longmapsto Df(a)$. We identify $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ with the Euclidean space $\mathbb{R}^{mn}$ as usual. We say f is of class C^1 or f is C^1 if the map $a \longmapsto Df(a)$ as above is continuous.

Inductively, we define f to be C^r if Df is of class C^{r-1} ($C^0 \equiv$ continuous maps).

→ If f is C^r $\forall\, r \geq 1$, we say f is C^∞ or smooth.

Exercise :- $f \in C^r \Rightarrow$ all mixed partials

$$D^\alpha f_i := \frac{\partial^{i_1} \partial^{i_2} \cdots \partial^{i_n}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} f_i$$

exist and are continuous $\forall\, 1 \leq i \leq m$ & $\forall\, \alpha = (i_1, \dots, i_n)$ with $|\alpha| := \sum_j i_j \leq r$; Conversely this $\Rightarrow f \in C^r$.

<u>Diffeomorphisms</u> :- Fix $1 \leq r \leq \infty$. A C^r map $f: U \to V$, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, open, is called a <u>C^r-diffeomorphism</u> if $\exists$ a C^r map $g: V \to U$ such that $g \circ f = 1_U$, $f \circ g = 1_V$. A C^∞-diffeom is called smooth. When $r=0$, the notion simply means f is a homeomorphism.

→ If $f: U \to V$ is a C^r diffeom, then the Jacobian matrix of f is invertible $(r \geq 1)$ pointwise (Why?). Note that if $\exists$ a C^r-diffeom : $U \to V$, then $n = m$. This also holds when $r=0$. This follows from "Brouwer's invariance of domain" thm from algebraic topology.

<u>Exercise</u> :- Let $r \geq 1$ and $f \in C^r; U \to V$ and assume f is a C^1-diffeom. Then f is a C^r-diffeom.

<u>Local diffeomorphisms</u> :- Let $U \subseteq \mathbb{R}^n$ open & $f: U \to \mathbb{R}^n$ be a C^r map. We call f a C^r-local diffeom at $p \in U$ if $\exists$ an open nbd W of a such that $f|_W : W \to f(W)$ →

is a C^r-diffeom.

<u>Remark</u>: If f is a C^r-local diffeom for all $p \in U$, then also f has invertible Jacobian at all $p \in U$; here $r \geq 1$.

<u>Example</u> :- Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x, y) \longmapsto (e^x \cos y, e^x \sin y).$$

(Equivalently $z \longmapsto e^z : \mathbb{C} \to \mathbb{C}$). This is <u>not</u> a diffeom (Why?), but f restricted to any open strip $\mathbb{R} \times (b, b+2\pi)$ is a diffeom. Note that f maps this open strip (diffeomorphically) to the <u>half-slit plane</u>, i.e. $\mathbb{R}^2$ minus the half ray $\{(\lambda \cos b, \lambda \sin b); \lambda \geq 0\}$ (picture this?).

<u>Exercise</u> :- $U, V \subseteq \mathbb{R}^n$ be open, $f : U \to V$ a smooth bijection. Then f is a smooth diffeo $\Leftrightarrow$ f is a C^1-local diffeo at all $p \in U$.

<u>Inverse function theorem</u> :- Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be C^r, $r \geq 1$. Let $p \in U$ and $Df(p)$ be invertible. Then f is

a C^r-local diffeom at p.

<u>(Sketch of) proof</u> :- First suppose f is an <u>affine map</u>, i.e. $f(x) = Ax + v$ where $A: \mathbb{R}^n \to \mathbb{R}^n$ is <u>linear</u> and $v \in \mathbb{R}^n$ is a <u>fixed</u> vector. Then, for $\omega, z \in U$, we have

$$f(\omega) - f(z) = A\omega - Az = Df(z)(\omega - z) \quad (?).$$

Hence if $Df(z) = A$ is invertible and we wish to have the <u>inverse image</u> ω of some $y = f(\omega)$, we can solve it as

$$\boxed{\omega = z - Df(z)^{-1}(f(z) - y)} \sim\sim \#$$

→ Let f be any map satisfying the hypothesis. Using $\varphi(x) := Df(p)^{-1}(f(x+p) - f(p)) : U \to \mathbb{R}^n$ we may assume (without loss of generality) that $p = 0$, $f(0) = 0$ and $Df(0) = I_{\mathbb{R}^n}$.

→ Let $B = \overline{B(0,\delta)}$ be the closed ball of radius δ around the origin s.t. $\overline{B} \subseteq U$ (δ to be chosen suitably). For $y \in \mathbb{R}^n$ define $T_y : B \to \mathbb{R}^n$ by $z \mapsto z - (f(z) - y)$, motivated by #, so that this would give the inverse image of y when f is affine.

Lemma-1 :- Let $U \subseteq \mathbb{R}^n$ be an open ball around the origin, $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and suppose $\exists\, M > 0$ such that $\left| \partial f_i / \partial x_j (x) \right| \leq M$ $\forall\, x \in U$. Then $\| f(x) - f(y) \| \leq n^2 M \, \| x - y \|$ $\forall\, x, y \in U$.

Lemma-2 (Contraction mapping principle).
Let (X, d) be a complete metric space, $f : X \to X$ be a contraction, i.e., $d(f(x), f(y)) \leq \lambda\, d(x, y)$ for some $0 \leq \lambda < 1$, $\forall\, x, y \in X$.
Then $\exists!\, x \in X$ such that $f(x) = x$.

The function $g : \mathbb{R}^n \to \mathbb{R}^n$, $g(x) = x - f(x)$.
Then $g(0) = 0$ & $Dg(0) = 0$; g is C^1
and $\exists\, \delta > 0$ s.t.

$$\left| \frac{\partial g_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(0) \right| < \frac{1}{2n^2} \quad \forall\, x \in \overline{B(0, \delta)}$$

$$\parallel$$

$$\left| \partial g_i / \partial x_j (x) \right|. \qquad \text{Use Lemma-1}$$

& put $y = 0$ to get $\| g(x) \| < \frac{\|x\|}{2}$ $\forall\, x \in \overline{B(0, \delta)}$

$\therefore\; g\left(\overline{B(0, \delta)} \right) \subseteq B(0, \delta/2)$.

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→ $T_y(x) = x - (f(x) - y) = y + g(x)$.

$\Rightarrow \|T_y(x)\| \leq \|y\| + \|g(x)\|$. Hence

for $y \in \overline{B(0, \delta/2)}$, $\|T_y(x)\| \leq \delta/2 + \delta/2 = \delta$

and so $T_y : \overline{B(0,\delta)} \to \overline{B(0,\delta)}$ and

$T_y(x) = x \iff y + x - f(x) = x \iff y = f(x)$.

→ f is 1-1 on $\overline{B(0,\delta)} \iff T_y$ has a !

fixed point in $\overline{B(0,\delta)}$.

→ For $y \in B(0, \delta/2)$, T_y is a contraction.

→ Hence f is 1-1 on $\overline{B(0,\delta)}$.

→ $f(\underbrace{B(0, \delta/4)}_{W})$ is open; $f^{-1} : f(W) \to W$

is smooth: $\forall p \in W$, $f \in C^1 \Rightarrow$

$$f(x) - f(p) = Df(p)(x-p) + \epsilon(x,p),$$

where $\lim_{x \to p} \epsilon(x,p)/\|x-p\| = 0$. (*)

Hence if $y = f(x)$ & $q = f(p)$, $A = Df(p)^{-1}$,

we have ⟶

$$A(y-q) = f^{-1}(y) - f^{-1}(q) + A\epsilon(f^{-1}(y), f^{-1}(q)).$$

Hence $f^{-1}(y) - f^{-1}(q) = A(y-q) + \epsilon'(y,q)$.

Now $\lim\limits_{y\to q} \dfrac{\epsilon'(y,q)}{\|y-q\|} = -A \lim\limits_{y\to q} \underbrace{\dfrac{\epsilon(f^{-1}(y), f^{-1}(q))}{\|y-q\|}}_{\to 0\ (?)}$ and

$\lightning$

Hence f^{-1} is differentiable $\forall\, p \in W$ 4

$(Df^{-1})(p) = (Df)(f^{-1}(p))^{-1}$ $\boxtimes$

- The IFT controls the <u>local</u> behaviour of a function, i.e. says what happens in a nbd of a point, from the information on its derivative at that point.
- It is possible (?) for a map to be a local diffeom at each point without being a global diffeom. However any $f: \mathbb{R} \to \mathbb{R}$ that is a C^r-local diffeom at each point (here $r \geq 1$) is indeed a global diffeo (?).

<u>Exercise</u>: 1. In general $C^{r+1} \subsetneqq C^r$.

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$ has non-vanishing derivative $Df(0) = 1/2$ at $x=0$. $\to$

However, f is <u>not</u> a diffeom in <u>any</u> nbd of $x=0$; in fact f even fails to be 1-1 in any nbd of 0. This shows that f being at least C^1 in IFT cannot be relaxed.

One uses IFT to prove the

<u>Implicit function Theorem</u>: Let $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable in an open set containing $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ and $f((a,b)) = 0$. Let M be the $m \times m$ matrix $\left(\dfrac{\partial f_i}{\partial x_{n+j}}(a,b)\right)_{1 \le i,j \le m}$.

If $\det M \neq 0$, there exists an open set $V \subseteq \mathbb{R}^n$, $a \in V$; an open $W \subseteq \mathbb{R}^m$, $b \in W$, such that for every $x \in V$, $\exists!\ g(x) \in W$ (with $g(a) = b$), $f(x, g(x)) = 0 \ \forall\, x \in V$.

The function $g: V \to W$ thus defined is differentiable. $\boxtimes$

In the next class, we'll see some applications of these results. It cannot be emphasized enough that these results are very fundamental to differential geometry.