Lecture - 4

Remarks on the Implicit Function Theorem :- To understand what the above achieves, consider a relation, say in $\mathbb{R}^2$, given as $F(x,y)=0$. Such a rel^n does not necessarily represent a function (of say x), i.e. this may not lead to a solution of the form $y=f(x)$ for a function f (e.g. $F(x,y)=x^2+y^2-3$).

One naturally asks: When, a rel^n $F(x,y)=0$, represents a function, or equivalently, when can the eq^n $F(x,y)=0$ be solved explicitly for y in terms of x, giving a ! solution?

The Implicit function theorem tells us that 'locally' this can be achieved (for good F), i.e. given a point (x_0,y_0) with $F(x_0,y_0)=0$, under 'certain conditions', $\exists$ a nbd of (x_0,y_0) such that in this nbd, $F(x,y)=0$ yields a functional sol^n $y=f(x)$. The conditions stipulate that F & $\partial F/\partial y$ be continuous in some nbd of (x_0,y_0) and $\frac{\partial F}{\partial y}(x_0,y_0)\neq 0$.

→ **More generally**: let $f:\mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$ be cont. diff. in an open nbd of $(a,b)\in\mathbb{R}^n\times\mathbb{R}^m$, $f((a,b))=0$. Assume that the $m\times m$ matrix →

$\left(\frac{\partial f_i}{\partial x_{n+j}}(a,b)\right)_{1\le i,j\le m}$ is invertible. Then $\exists$ a nbd $A\times B \subseteq \mathbb{R}^n \times \mathbb{R}^m$ of (a,b) and a diff. function $g: A \to B$, $g(a)=b$, s.t.

$f(x, g(x)) = 0 \ \forall \ x \in A.$ $\boxtimes$

Submersion: A C^r-map, for $r \geq 1$, $f: \underset{\subseteq \mathbb{R}^n}{U} \to \mathbb{R}^m$ is a submersion at $a \in U$ (open) if $Df(a)$ is surjective. (This $\Rightarrow n \geq m$ and Jacobian of f at a has rank m).

Immersion :- Let $r \geq 1$. A C^r-map $f: U \to \mathbb{R}^m$ is a C^r-immersion at $a \in U$ if $Df(a)$ is injective; (so in this case $n \leq m$ and the Jacobian of f at a has rank n).

Rank theorem :- Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be cont. diff. in an open set $\ni a$, $n \geq m$. If $f(a)=0$ and the $m\times n$ matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)$ has rank m; then $\exists$ an open set $a \in A \subseteq \mathbb{R}^n$ and a diff. func^n $h: A \to \mathbb{R}^n$ with a diff inverse (i.e. h is a diffeom) s.t.

$f \circ h(x_1, \ldots, x_n) = (x_{n-m+1}, \ldots, x_n)$. $\boxtimes$

Hence one can modify the coordinates around the point a by the function h to get a simpler form for f locally.

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Level sets: Let $U \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times 1$ be open & $f: U \to \mathbb{R}$ be smooth. Let $c \in \mathbb{R}$ be such that for $S = f^{-1}(c)$, $(\nabla f)(p) \neq 0$ $\forall\, p \in S$ (we assume such c exists for f). We then call S a hypersurface in $\mathbb{R}^{n+1}$.

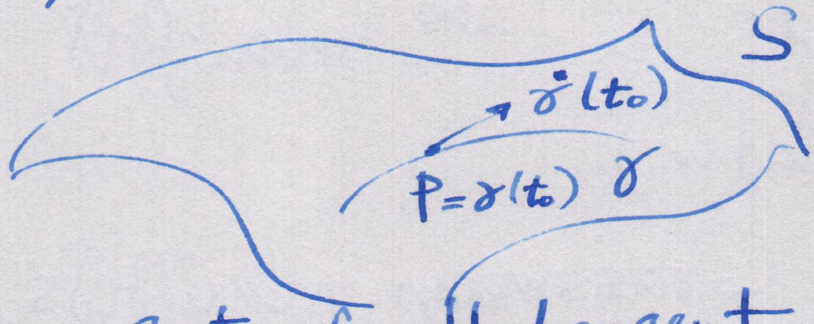
A parameterized curve in $\mathbb{R}^{n+1}$ is a smooth function $\gamma: I \to \mathbb{R}^{n+1}$, I some open interval in $\mathbb{R}$; so $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+1}(t))$, $\gamma_i : I \to \mathbb{R}$ are smooth. Denote

$$\dot\gamma(t) := (\dot\gamma_1(t), \dots, \dot\gamma_{n+1}(t)),$$

where $\dot\gamma_i(t) = \frac{d}{dt}\gamma(t)$. When $\dot\gamma(t_0) \neq 0$, we call $\dot\gamma(t_0)$ the tangent vector to $\gamma(t)$ at the point $\gamma(t_0)$. (justify this?).

We define, for S as above ⟶

a <u>tangent vector to S at</u> $p \in S$ is, by def^n, $\dot\gamma(t_0)$ for some curve $\gamma \subseteq S$, where $p = \gamma(t_0)$, i.e. γ lies on S and passes through p.



We denote $T_p(S)$ = set of all tangent vectors to S at p. We first observe

<u>Lemma</u> :- The <u>gradient</u> of f at p, i.e. $\nabla f(p)$ $(\equiv Df(p))$ is <u>orthogonal</u> to $T_p(S)$.

<u>Proof</u> :- Let $v \in T_p(S)$. Then $v = \dot\gamma(t_0)$ for some $\gamma : I \to S = f^{-1}(c)$. Therefore $f(\gamma(t)) = c \;\; \forall\; t \in I$. Differentiating we get $\frac{d}{dt}(f \circ \gamma) = 0$, so $Df(\gamma(t_0)) \cdot \dot\gamma(t_0)$

$$= \nabla f(p) \cdot \dot\gamma(t_0) = 0$$

$$= \langle \nabla f(p), \dot\gamma(t_0) \rangle . \qquad \boxtimes$$

An application of the Implicit func^n thm: $\longrightarrow$

<u>Genralized level sets</u>: We'll discuss level sets for $f: U^{\subseteq \mathbb{R}^{n+k}} \to \mathbb{R}^k$, for smooth f and <u>define</u> the <u>tangent space</u> T_pS of $S = f^{-1}(0)$ at $p \in S$ as the <u>set</u> of all vectors $v \in \mathbb{R}^{n+k}$ such that $v = \dot{\gamma}(0)$ for a <u>smooth</u> $\gamma: I \to S$, $\gamma(0) = p$, $0 \in I$. We sketch a proof of

<u>Theorem</u>: Let $f: U \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^k$ be C^1. Assume $S := f^{-1}(0) \neq \emptyset$ and $\forall p \in S$, $Df(p)$ has rank k. Then $T_p(S) = \operatorname{Ker} Df(p)$. In particular $T_p(S)$ is an n-dim'l vector space$/\mathbb{R}$.

<u>Proof (sketch)</u>: • Just like the case $k=1$, $T_p(S) \subset \operatorname{Ker}(Df(p))$ (Chain rule).

• Conversely, let $\omega \in \mathbb{R}^{n+k}$ be such that $Df(p)(\omega) = 0$. Since $Df(p)$ has rank k, by composing f with a suitable permutation, we may assume $\left(\frac{\partial f_i}{\partial x_{n+j}}\right)_{1 \le i,j \le k}$ is invertible. $\rightsquigarrow$

4-6. Write any $z \in U \subseteq \mathbb{R}^{n+k}$ as $z = (x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$; $p = (a, b)$, $w = (u, v)$.

Let $G : U \subseteq \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$ be defined by

$G(x, y) := (x, f(x, y))$. Then

- $DG(p)$ is invertible (?). So $\exists$ nbds U_1 of p in $\mathbb{R}^{n+k}$ and V of a in $\mathbb{R}^n$ and a C^1 map $h : V \to \mathbb{R}^k$ such that
- $h(a) = b$; $\{(x, y) \in U_1 \mid f(x, y) = 0\}$

$$= \{(x, h(x)) \mid x \in V\}.$$

- Consider $\gamma(t) := a + tu$ $\quad (w = (u, v))$.

$\Rightarrow$ $\gamma(t) \in V$ for sufficiently small t.

- $c(t) := G^{-1}((\gamma(t), 0))$. Then c is C^1 as G is ; $c(0) = G^{-1}((\gamma(0), 0))$

$$= G^{-1}((a, 0)) = (a, b) = p.$$

- $\dot{c}(0) = w$. This follows from the computation $DG(p)(w = (u, v)) = (u, 0)$.

$\boxtimes$

4-7.

Given $f: U \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^k$ as in the theorem and $S = f^{-1}(0)$ the corresponding level set, for $p \in S$, $Df(p): \mathbb{R}^{n+k} \to \mathbb{R}^k$ (rank k) $\Rightarrow$ $\ker(Df(p)) = T_p(S)$.

When $k=1$, we have $T_p(S) = \nabla f(p)^{\perp} \subseteq \mathbb{R}^{n+1}$

and $\nabla f(p)$ is the normal to the hypersurface S, $\forall\, p \in S$.

Towards a differentiable structure on S:

Let $S = f^{-1}(0)$, $f: \mathbb{R}^{n+1} \to \mathbb{R}$ a C^1-map with $\nabla f(p) \neq 0$ $\forall\, p \in S$ (the rank cond^n.)

Fix $p_0 = (a_1, \dots, a_{n+1}) \in S$, so $\nabla f(p_0) \neq 0$.

Let $Q_0 = (a_1, \dots, a_n) \in \mathbb{R}^n$ be the proj^n of p_0 onto $\mathbb{R}^n$. By the Implicit func^n thm, $\exists$ an open set $V_0 \ni Q_0$, an interval $(a_{n+1} - \delta, a_{n+1} + \delta)$ and a C^1-map g on V_0 such that $\cdot\, f(x_1, \dots, x_n, g(x_1, \dots, x_n)) = 0$ in V_0 $\rightsquigarrow$

- $a_{n+1} = g(a_1, \dots, a_n)$, $|a_{n+1} - g(x_1, \dots, x_n)| < \delta$ in V_0.
- Any sol^n $(x_1, \dots, x_{n+1}) \in V_0 \times (a_{n+1}-\delta, a_{n+1}+\delta)$ of the eq^n $f(x_1, \dots, x_{n+1}) = 0$ is of the form $x_{n+1} = g(x_1, \dots, x_n)$.

Let $U_0 = S \cap V_0 \times (a_{n+1}-\delta, a_{n+1}+\delta)$. Then $P_0 \in U_0$ and the restriction of the projection : $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ to U_0 gives

$$\varphi_0 : U_0 \longrightarrow V_0 \,,\ \varphi_0(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n) \in V_0 \,,$$

with $\varphi_0^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_n, g(x_1, \dots, x_n))$.

- φ_0 is a diffeom.

<u>Thm</u> :- Let $f : \underset{U \subseteq}{\mathbb{R}^n} \xrightarrow{n \geq m} \mathbb{R}^m$ be smooth, $S = f^{-1}(0) \neq \phi$, the Jacobian of f has rank m at all pts of U. Then every point of S has an open nbd diffeomorphic to an open subset of $\mathbb{R}^{n-m}$.

Proof is an exercise for you. Next class : Important examples of S as above.