Lecture-5

Examples :- 1. Let $S = \{ X \in M_n(\mathbb{R}) \mid X^t = X \}$. Then S is a vector space $/\mathbb{R}$ of dim $n(n+1)/2$. Let S be identified with $\mathbb{R}^{n(n+1)/2}$ and $f : \mathbb{R}^{n^2} \equiv M_n(\mathbb{R}) \to \mathbb{R}^{n(n+1)/2} \equiv S$ be defined by
$$f(X) = XX^t.$$
We are interested in
$$O(n, \mathbb{R}) := f^{-1}(I);$$
the orthogonal group $= \{ X \in M_n(\mathbb{R}) \mid XX^t = I \}$. We'll check the rank condition on f. We have
$$Df(A)(X) = \lim_{t \to 0} [f(A+tX) - f(A)]$$
$$\lim_{t\to 0} \frac{1}{t}\left[(A+tX)(A+tX)^t - AA^t\right] = AX^t + XA^t.$$
for $A \in O(n)$

Claim: $DF(A)$ has rank $n(n+1)/2$; To see this, we must show that for $Y \in S$, $\exists\, X \in M_n(\mathbb{R})$ with $AX^t + XA^t = Y = Y^t$. If we find X with $AX^t = Y/2$, then $Y^t = Y \Rightarrow X$ would give a solution to $AX^t + XA^t = Y$. Now $AX^t = Y/2$ $\Rightarrow X = YA/2$ is a solution. Hence $Df(A)$ has full rank $\forall A \in O(n)$. By the theorem done in the last class, $O(n)$ is a 'manifold' of dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

We also note the computation: $DF(I)(X) = X + X^t$;

So $\operatorname{Ker}(DF(I)) = \{X \in M_n(\mathbb{R}) \mid X + X^t = 0\}$

$$= \operatorname{Skew}_n(\mathbb{R}) = \{X \in M_n \mid X^t = -X\}.$$

Hence $T_I(O(n)) = \operatorname{Skew}_n(\mathbb{R})$, the 'tangent space at I' to $O(n)$; $\overset{\shortparallel}{o}(n) := T_I(O(n))$.

2. Consider $f : M_n(\mathbb{R}) \to \mathbb{R}$, $f(A) = \det(A)$.

Then $f^{-1}(1) =: SL_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \det X = 1\}$.

For $X = (x_{ij})_{1 \le i,j \le n}$, $\det X$ is a polynomial in x_{ij}'s, hence $f \in C^\infty$ (?). We will compute $\nabla f(A)$ for $A \in SL_n(\mathbb{R})$. We note that $\det X = x_{11} \det X_{11} - x_{12} \det X_{12} + \cdots$

$$+ (-1)^{n+1} x_{1n} \det X_{1n}$$

Where X_{1j} is the submatrix of X obtained by deleting the 1^{st} row & j^{th} column.

In particular, X_{1j} involves variables except those in the 1^{st} row, i.e. x_{1k}'s. Hence

$\dfrac{\partial f}{\partial x_{11}} = \det X_{11}$, and therefore

$$\frac{\partial f}{\partial x_{11}}(I) = 1 \neq 0.$$

Let $A_0 \in SL_n(\mathbb{R})$ be arbitrary. Define $y_{ij} = ij^{th}$ entry of $A_0^{-1}X$. So, for $X = A_0$ we get $Y = (y_{ij}) = I$ and $f(X) = f(A_0 Y) = \det(A_0 Y) = \det(A_0) f(Y)$.

We have therefore $1 = \frac{\partial f}{\partial y_{11}}(I) = \sum_{i,j} \frac{\partial f}{\partial x_{ij}}(A_0) \frac{\partial x_{ij}}{\partial y_{11}}$.

Hence for some (i,j), we must have $\frac{\partial f}{\partial x_{ij}}(A_0) \frac{\partial x_{ij}}{\partial y_{11}} \neq 0$ and so $\frac{\partial f}{\partial x_{ij}}(A_0) \neq 0$.

This shows $\nabla f(A_0) \neq 0$ and thus $SL_n(\mathbb{R})$ is a manifold of dim n^2-1, i.e. every matrix in $SL_n(\mathbb{R})$ has an open nbd $\simeq \mathbb{R}^{n^2-1}$!

→ To compute $T_I(SL_n(\mathbb{R}))$, we must compute $\ker(Df(I))$. We first define:

Multilinear maps :- A bi-linear map $B: V \times W \to Z$, V, W, Z vector spaces / a field k is a map such that it is linear in each variable :

- $B(a_1 v_1 + a_2 v_2, w) = a_1 B(v_1, w) + a_2 B(v_2, w)$ →

- $B(v, b_1 w_1 + b_2 w_2) = b_1 B(v, w_1) + b_2 B(v, w_2)$.

More generally we define an r-linear map $B : V_1 \times \cdots \times V_r \to W$, linear in each variable.

Lemma:- Let $f : V_1 \times \cdots \times V_r \to W$ be a multi-linear map of Euclidean spaces (so $V_i = \mathbb{R}^{n_i}$, $W = \mathbb{R}^m$ etc)

Then f is differentiable with

$$Df(v_1, v_2, \ldots, v_r)(x_1, \ldots, x_r) = \sum_{i=1}^{r} f(v_1, \ldots, v_{i-1}, x_i, v_{i+1}, \ldots, v_r) \qquad (?)$$

Proof:- We have $Df(v_1, \ldots, v_r)(x_1, \ldots, x_r)$

$$= \lim_{t \to 0} \frac{1}{t} \left[f((v_1 + t x_1, \ldots, v_r + t x_r)) - f((v_1, \ldots, v_r)) \right]$$

We first expand in powers of t

$$\cancel{= \lim_{t \to 0} \frac{1}{t} \left[f(t x_1, v_2, \ldots, v_r) + f(v_1, v_2 + t x_2, \right.}$$

$$= f(v_1, v_2, \ldots, v_r) + f(t x_1, v_2, \ldots, v_r) + f(v_1, t x_2, v_3, \ldots, v_r) + \cdots + f(v_1, \ldots, v_{r-1}, t x_r)$$

+ terms involving higher ~~pow~~ number of coordinates with t. Hence we have (?)

$$= \lim_{t \to 0} \frac{1}{t} \left[t f(x_1, v_2, \ldots, v_r) + t f(v_1, x_2, v_3, \ldots, v_r) + \cdots + t f(v_1, \ldots, v_{r-1}, x_r) \right]$$

$$= \sum_i f(v_1, \dots, x_i, \dots, v_r). \qquad \boxtimes$$

<u>Back to $SL_n(\mathbb{R})$</u> :- We think of $M_n(\mathbb{R})$ as $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n}$, vectors in $\mathbb{R}^n$ written column-wise, so $(e_1, \dots, e_n) = I$. We apply the lemma to $\det : M_n(\mathbb{R}) = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

We then have

$$D \det(I)(x_1, \dots, x_n)$$

$$= \sum_{i=1}^{n} \det(e_1, \dots, x_i, \dots, e_n).$$ We see (?) that

$$\det(e_1, \dots, x_i, \dots, e_n) = \det \begin{pmatrix} 1 & 0 & \cdots & x_{1i} & \cdots & 0 \\ 0 & 1 & \cdots & x_{2i} & & 0 \\ \vdots & 0 & & \vdots & & \vdots \\ \vdots & \vdots & \cdots & x_{ii} & & \vdots \\ 0 & 0 & \cdots & x_{ni} & & 1 \end{pmatrix}$$

$$= x_{ii}.$$

Therefore $D\det(I)(X) = \operatorname{trace}(X)$.

We thus get $T_I(SL_n(\mathbb{R})) = \operatorname{Ker}(X \mapsto \operatorname{trace}(X))$

$= \mathfrak{sl}_n(\mathbb{R}) =$ $n \times n$ trace 0 matrices$/\mathbb{R}$.

<u>Exercises</u> :- 1. Compute $D\det(A)(X)$ for: $\det : GL_n(\mathbb{R}) \to \mathbb{R}$

2. Let $U(n) = \{X \in GL_n(\mathbb{C}) \mid X\bar{X}^t = I\}$

5-6 Where $X^* := X\bar{X}^t$, $\bar{X} = (\bar{x}_{ij})$. Consider $f: \mathbb{C}^{n^2} \to H$ be the map $X \mapsto XX^*$, where $H = \{X \in M_n(\mathbb{C}) \mid X = X^*\}$, the $\mathbb{R}$-vector space of $n\times n$ hermitian = self-adjoint matrices.

(i) Compute $\dim_{\mathbb{R}} H$. (ii) Realize $U(n)$ as $f^{-1}(I)$, check appropriate conditions on f required for $U(n)$ to be a level set for f as in the case of $O(n)$. (iii) Compute $T_I(U(n))$ as the set $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$; compute $\dim U(n)$ as a <u>real</u> manifold. (iv) Compute the same for $SU(n) = \{X \in U(n) \mid \det X = 1\}$ and show $T_I(SU(n)) = \mathfrak{su}(n) = \{X \in \mathfrak{u}(n) \mid \operatorname{trace}(X) = 0\}$.

The above discussion describes what are called the 'classical' Lie groups and their 'Lie algebras' as a vector space = the tangent space at the identity element. We'll discuss these in general later.

Remark 1. The groups SL_n, SO_n, SU_n are respectively called the special linear, special orthogonal and the special unitary group, while GL_n, O_n, U_n are called (resp.) the general linear, orthogonal and the unitary group.

2. Class C^ω: $f: \underset{\text{open}}{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^∞. We call f as real analytic in a nbd of $p \in U$ if $\exists$ $r > 0$ such that the series

$$\sum_{i_1 \geq 0, \dots, i_n \geq 0} \frac{1}{i_1! \cdots i_n!} \left[\frac{\partial^{i_1} \partial^{i_2} \cdots \partial^{i_n} f}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} (p) \right] (x_1 - p_1)^{i_1} \cdots (x_n - p_n)^{i_n}$$

converges absolutely to $f(x_1, \dots, x_n)$ for $\|x - p\| < r$; $p = (p_1, \dots, p_n)$, $x = (x_1, \dots, x_n)$.

The above series is the Taylor series of f at p.

- A function that is real analytic at all $p \in U$ is called a real analytic or C^ω function on U. A map $f: \underset{\text{open}}{U \subseteq \mathbb{R}^n} \to \mathbb{R}^m$ is C^ω if each coordinate function of f is such.
- A function may well be C^∞ (i.e. smooth) yet not C^ω; e.g. $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto e^{-1/x^2}$, $x \neq 0$ & $0 \mapsto 0$ (?).

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Taylor's theorem :- Let $f : U \underset{open}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ be C^k, $p \in U$ and assume for $x \in U$, the line segment $p + tx$, $0 \le t \le 1$ lies in U. Let $x^{(r)} = \underbrace{(x, \dots, x)}_{r}$ and $f^r(p)\, x^{(r)} := D^r f(p)(x, \dots, x)$; here $D^i f = D(D^{i-1} f)$; $D^{i-1} f : U \to \mathbb{R}^m$, $p \mapsto (D^{i-1} f)(p)$.

Then

$$f(p+x) = f(p) + \frac{f^1(p)}{1!}x + \dots + \frac{1}{k!} f^k(p)\, x^{(k)} + \varepsilon_k(x),$$

where $\varepsilon_k(x) / \|x\|^k \to 0$ as $x \to 0$.

Example :- Let $\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be the map $\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$. Let us discuss this a bit.

The exponential map :- • For $X = (x_{ij}) \in M_n(\mathbb{R}) = \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$, let $\|X\|_\infty = \max_{i,j} |x_{ij}|$. This is a norm on $M_n(\mathbb{R})$, equivalent to the operator norm $\|\;\|$ on $\mathbb{R}^n$. Then (Exercise)

- $\|AB\| \le \|A\| \|B\|$, $\|A^r\| \le \|A\|^r$
- A sequence $X_k \to X$ in the operator norm if and only if $x_{ij}^{(k)} \longmapsto x_{ij}$ $\forall\ 1 \le i, j \le n$ as $k \to \infty$.

Here $X_k = (x_{ij}^{(k)})$ & $X = (x_{ij})$ $\rightsquigarrow$

- If $\sum_{k=0}^{\infty} \|A_k\|$ is convergent, then $\sum_{k=0}^{\infty} A_k$ converges to an element A in $M_n(\mathbb{R})$.
- For any $X \in M_n(\mathbb{R})$ the series $e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$ is convergent & the sum is denoted by $\exp(X) \equiv e^X$.
- $X \longmapsto e^X$ is a real analytic map on $M_n(\mathbb{R})$.
- For $A, B \in M_n(\mathbb{R})$ with $AB = BA$, we have

$$e^{A+B} = e^A e^B = e^B e^A = e^{B+A}$$ (We'll discuss this later).

- For $A, B \in M_n(\mathbb{R})$, $A e^B A^{-1} = e^{ABA^{-1}}$.
- Fix $X \in M_n(\mathbb{R})$. Let $\gamma : \mathbb{R} \to GL_n(\mathbb{R})$ (?) be defined by $\gamma(t) = e^{tX}$. Then $\dot{\gamma}(t) = X \cdot \gamma(t)$. So $\gamma(0) = I$, $\dot{\gamma}(0) = X$.
- $e^{tX} \cdot e^{-tX} = I \implies e^{tX}$ is invertible $\forall t$, and all X. So indeed $X \mapsto e^X \in GL_n(\mathbb{R})$.

— We'll discuss more on this. The exponential map provides an important tool in study of Lie groups and their 'Lie algebras'.