Lecture-6

Differentiable / smooth / C^r / C^ω - manifolds :- In DG-1 you saw geometry of curves and surfaces and these were sub-structures of $\mathbb{R}^2$ or $\mathbb{R}^3$ mostly, where we had the coordinates from the ambient Euclidean space available to us in order to do analysis on these structures; these are needed to talk of differentiability, to make sense of 'direction' on such structures, or to talk of normal at a given point. We wish to treat curves/surfaces now in their own right as geometric objects and talk of differentiable functions etc. on them. The idea is to look at these 'locally' and use coordinates from Euclidean spaces suitably to do calculus. We first define:

Def^n: (Topological manifold): A manifold of dimension n is a Hausdorff topological space M such that every $p \in M$ has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$.

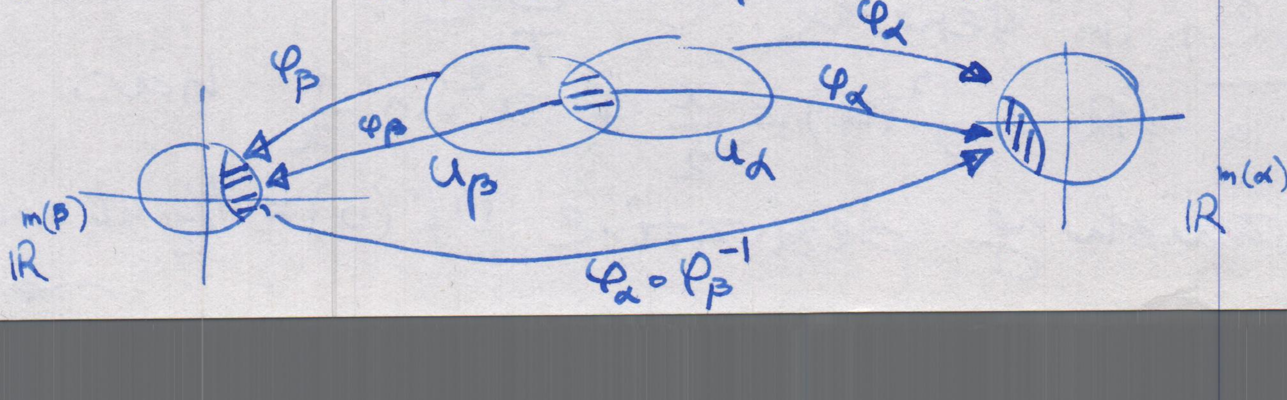
A local chart is a pair (U, φ), $U \subseteq M$ open and $\varphi : U \xrightarrow{\sim} V \underset{\text{open}}{\subseteq} \mathbb{R}^n$, a homeomorphism.

- A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ of local charts

such that $\bigcup_{\alpha\in\Lambda} \mathcal{U}_\alpha = M$, is called an <u>atlas.</u>
$(U_\alpha, \varphi_\alpha)$ is also called a coordinate chart and for $p \in \mathcal{U}_\alpha$, $\varphi_\alpha(p)$ has coordinates in $\mathbb{R}^n$, which are called <u>coordinates of p</u> with respect to $(U_\alpha, \varphi_\alpha)$.

A <u>more general</u> def of a (topological) manifold: M Hausdorff, with a covering (open) $\{U_\alpha\}_{\alpha\in\Lambda}$, homeomorphisms $\varphi_\alpha : U_\alpha \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^{m(\alpha)}$;
but then we do not define dimension,
e.g. $M = \underset{\mathbb{R}}{\overline{\qquad\qquad}} \sqcup S^2$.

When we wish to talk of dimension of M, we either work with the first def^n, or <u>demand</u> $(M, \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda})$ satisfies the <u>C^k- compatibility condition</u>; $\exists\, k \in \mathbb{N} \cup \{\infty\}$ such that whenever $U_\alpha \cap \mathcal{U}_\beta \neq \varphi$, the map
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
(these are Euclidean open sets) is C^k



Remarks: 1. Suppose on a topological manifold M, (U,φ) and (V,ψ) are two charts, $U\cap V\neq\varphi$. For $x\in U\cap V$, we have two sets of coordinates: the φ-coordinates $\{\varphi_1(x),\dots,\varphi_{m(\varphi)}(x)\}$ and the ψ-Coordinates $\{\psi_1(x),\dots,\psi_{m(\psi)}(x)\}$. These, in general would be different. The transition homeomorphisms $\psi\circ\varphi^{-1}:\varphi(U\cap V)\to\psi(U\cap V)$ & $\varphi\circ\psi^{-1}:\psi(U\cap V)\to\varphi(U\cap V)$, express the ψ-coords and φ coordinates in terms of the ψ-coords and vice-versa. When M is a C^k-manifold, i.e. $\{(U_\alpha,\varphi_\alpha)\}_\alpha$ is a C^k-atlas, the transition maps are C^k-diffeomorphisms.

2. If we assume M to be connected, then, by the inverse function theorem, $m(\alpha)=m(\beta)$ $\forall\,\alpha,\beta\in\Lambda$ (discuss), so we can define the dimension of M as the common value $m = m(\alpha)$, $\alpha\in\Lambda$.

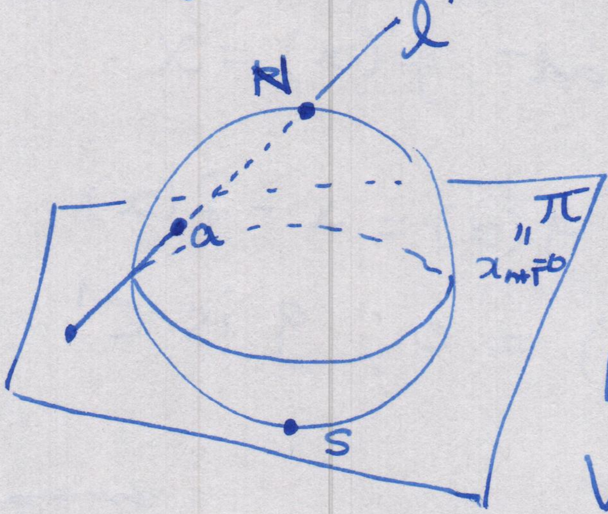
3. We call an atlas $\mathcal{A}$ on M as smooth or C^∞ if it is C^k $\forall\, k\in\mathbb{N}$.

4. We define analytic (or C^ω) manifolds analogously.

Examples :- 1. Let $U \overset{i}{\subseteq} \mathbb{R}^n$ be an open subset, i the inclusion map. Then U is a C^∞-manifold (in fact C^ω) of dimension n. The C^ω-atlas $\{(U, i)\}$ does the job in this case.

- Consider $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R}) \equiv \mathbb{R}^{n^2}$ as an open subset; so $GL_n(\mathbb{R})$ is an n^2-dim'l real manifold, of smooth or C^ω type.

2. The n-Sphere S^n: $S^n = \{ a \in \mathbb{R}^{n+1} \mid \|a\| = \left(\sum_{i=1}^{n+1} a_i^2\right)^{1/2} = 1 \}$ has the subspace topology from $\mathbb{R}^{n+1}$.



Let $N = (0, 0, \cdots, 0, 1) \in S^n$ be the 'north pole' and $S = (0, 0, \cdots, 0, -1)$ the 'south pole' on S^n.
Let $U = S^n - \{N\}$ and $V = S^n - \{S\}$. Then U, V are open in S^n, $U \cup V = S^n$. We define the coordinate functions on U & V to give a C^∞-atlas as follows: this is the 'stereographic projection'; for $a \in U$, let l be the line by a and N and π be the plane $\subseteq \mathbb{R}^{n+1}$ given by $x_{n+1} = 0$. Define $\varphi(a)$ = the point of intersection of l and π; $\varphi : U \to \mathbb{R}^n = \{x_{n+1} = 0\}$ is given by $\varphi((a_1, \cdots, a_{n+1})) = (x_1, \cdots, x_n)$, where $x_i = a_i / (1 - a_{n+1})$

for $1 \le i \le n$ (note $a_{n+1} \neq 1$ as $a \in U$). Similarly, $\psi: V \to \mathbb{R}^n$ is given by $a = (a_1, \dots, a_{n+1}) \longmapsto (y_1, \dots, y_n)$ where $y_i = a_i / _{1 + a_{n+1}}$, $1 \le i \le n$.

<u>Check</u> :- $\varphi\psi^{-1}$ & $\psi\varphi^{-1}$ are C^∞. So $(S^n, \{(U,\varphi), (V,\psi)\})$ is a C^∞-manifold.

(for $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $(x_1, \dots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} x_i^2 - 1$, $Df(p) \neq 0$ for $p \in S^n = f^{-1}(0)$ and the Implicit function thm applies to get a nbd for $p \in S^n$, projecting onto the 'plane' $x_{n+1} = 0$, yielding a C^∞-atlas).

3. Level sets of smooth functions : $\mathbb{R}^n \to \mathbb{R}^m$ as we discussed, for well behaved functions, are C^k-manifolds. (Example 2.1.11 of Kumaresan).

4. <u>The real projective space $\mathbb{R}P^n \equiv P^n(\mathbb{R})$</u> :- We let $\mathbb{R}P^n$ = Set of lines through 0 in $\mathbb{R}^{n+1}$ = set of 1-dim'l subspaces of $\mathbb{R}^{n+1}$. Given a 1-dim'l subspace $\subseteq \mathbb{R}^{n+1}$, it is the set of all $\mathbb{R}$-multiples of a fixed <u>nonzero</u> vector. Any two <u>nonzero</u> vectors in a line differ by a scalar $\in \mathbb{R}$. To topologize $\mathbb{R}P^n$ we proceed as follows: On $\mathbb{R}^{n+1} \setminus \{0\}$, define the <u>equivalence</u>

$v \sim w \iff v = \lambda w$ for some $\lambda \in \mathbb{R} \setminus (0)$.

The an equivalence class $[v]$ can be identified with the line $\mathbb{R}\cdot v$; give $\mathbb{RP}^n \equiv \mathbb{R}^{n+1}\setminus\{0\}/\sim$ the quotient topology, i.e. $U \subseteq \mathbb{RP}^n$ is declared open if its "pull back" is open in $\mathbb{R}^{n+1}\setminus(0)$ (hence in $\mathbb{R}^{n+1}$) via the class map $\pi : \mathbb{R}^{n+1}\setminus(0) \to \mathbb{RP}^n$, $x \mapsto [x]$.

- π is an open map: to see this, for $t \in \mathbb{R}\setminus(0)$ define $f_t : \mathbb{R}^{n+1}-(0) \to \mathbb{R}^{n+1}-(0)$, $f_t(x) = tx$. Then f_t is a homeomorphism. Let $U' \subseteq \mathbb{R}^{n+1}\setminus(0)$ be open. Then $\pi(U') = \bigcup_{t\neq 0} f_t(U')$. Each $f_t(U')$ is open in $\mathbb{R}^{n+1}\setminus(0)$, hence $\pi(U')$ is open in $\mathbb{RP}^n$. Using this, check that $\mathbb{RP}^n$ is Hausdorff. Define U_i, $1 \leq i \leq n+1$, by

$$U_i := \{ [x] \in \mathbb{P}^n \mid x_i \neq 0,\ x = (x_1, \dots, x_i, \dots, x_{n+1}) \in \mathbb{R}^{n+1}\setminus(0) \}.$$

- Def^n of U_i makes sense, since for $\lambda \in \mathbb{R}\setminus(0)$ and a representative x of $[x] \in U_i$, we have $\lambda x = (\lambda x_1, \dots, \lambda x_i, \dots, \lambda x_{n+1})$ and $\lambda x_i \neq 0$.
- U_i are open in $\mathbb{P}^n$. Define $\varphi_i : U_i \to \mathbb{R}^n$ by $\varphi_i([y]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$ ↷

 where $(x_1, \ldots, x_{n+1})$ is an arbitrary elem. of $[y]$. Then φ_i is well defined (?).

- φ_i is continuous, 1-1 and onto (?).
- $\varphi_i \circ \varphi_j^{-1}$ is C^∞, where defined, for $1 \leq i, j \leq n+1$. Hence $\mathbb{R}P^n = \mathbb{P}^n_{\mathbb{R}} = P^n(\mathbb{R})$ is a smooth manifold of dim n. $(\bigcup_i U_i = \mathbb{P}^n)$.

<u>Exercise</u>:- $\mathbb{R}P^n \simeq S^n / \{\pm 1\}$; $x, y \in S^n$, $x \sim y \Leftrightarrow x = \pm y$.

<u>Example 5</u>: <u>Grassmann manifolds</u>: These generalize the construction of $\mathbb{P}^n$. Fix k, $1 \leq k \leq n$. Let $G(k,n)$ denote the set of all k-dim'l vector subspaces of $\mathbb{R}^n$. So $\mathbb{R}P^n = G(1, n+1)$. We proceed to give a C^∞-manifold structure on $G(k,n)$. Let $F(k,n)$ denote the set of all <u>k-frames</u> in $\mathbb{R}^n$; a k-frame in $\mathbb{R}^n$ is (an ordered) set X of k-lin. indep. vectors in $\mathbb{R}^n$,

$x_1 = (x_{11}, \ldots, x_{1n})$, $x_2 = (x_{21}, \ldots, x_{2n})$, $\ldots$, $x_k = (x_{k1}, \ldots, x_{kn})$. We think of X as a $k \times n$ matrix with rows $x_1, \ldots, x_k$.

→ $M_{k \times n}(\mathbb{R}) \equiv \mathbb{R}^{kn}$, is a smooth manifold. $F(k,n)$ is the set of rank k-matrices. →

→ $F(k,n)$ is an <u>open subset</u> of $\mathbb{R}^{kn}$:
$X \in M_{k\times n}(\mathbb{R})$ has rank $k \Leftrightarrow$ rows are lin. indep.
$\Leftrightarrow$ not all $k\times k$ minor determinants are zero. (supply details).
Hence $F(k,n)$ is a smooth manifold.

→ We have the natural map $F(k,n) \xrightarrow{\pi} G(k,n)$
$X \longmapsto$ subspace spanned by $\{x_1, .., x_k\} \subseteq \mathbb{R}^n$.

→ $X = (x_1, .., x_k)$ & $Y = (y_1, .., y_k)$ give the same subspace $\subseteq \mathbb{R}^n \Leftrightarrow Y = CX$ for a non singular $\equiv$ invertible $k\times k$ matrix C.
(Compare with $\mathbb{RP}^n$ construction).

→ Define $\sim$ on $F(k,n)$ by $Y \sim X$ iff $\exists\, C \in GL_k(\mathbb{R})$ with $Y = CX$. Then $G(k,n) = F(k,n)/\sim$; $\pi : F(k,n) \to F(k,n)/\sim$ $(= G(k,n))$ is the map we defined above. Give $G(k,n)$ the quotient topology from $F(k,n)$.

→ $[\; G(1, \mathbb{R}) = \mathbb{R}^{\times} = \mathbb{R} \setminus \{0\} \;]$.

→ π is an open map (Exercise).

<u>C^∞-compatible charts</u>: We'll use $k\times k$ sub-matrices of $X \in M_{k\times n}(\mathbb{R})$ for this. →

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Let $J=(j_1,\cdots,j_k)$ be an ordered subset of $(1,\cdots,n)$ and J' be its complementary subset e.g. $J=(1,2,\cdots,k)$, then $J'=(k+1,\cdots,n)$. Let X_J be the $k\times k$ sub-matrix (x_{ij_l}), $1\le i,l\le k$ of the $k\times n$ matrix X and $X_{J'}$ the complementary $k\times(n-k)$ submatrix obtained by deleting out the columns $j_1,\cdots,j_k$ of X. Let $\tilde{U}_J=\{X\in F(k,n)\mid X_J \text{ is nonsingular}\}$ & $U_J=\pi(\tilde{U}_J)\subseteq G(k,n)$. Then $\tilde{U}_J$ is open in $F(k,n)$, hence U_J is open in $G(k,n)$.

→ Each $Y\in\tilde{U}_J$ is equivalent to exactly one $k\times n$ matrix X in which $X_J=k\times k$ identity matrix

e.g. for $J(1,2,\cdots,k)$, $X=\begin{pmatrix}1&\cdots&0&x_{1,k+1}&\cdots&x_{1n}\\ \vdots&&\vdots&\vdots&&\vdots\\ 0&\cdots&1&x_{k,k+1}&\cdots&x_{kn}\end{pmatrix}$

- If $X\sim Y$ and Y_J is non-singular, then $X=Y_J^{-1}Y$. Define $\varphi_J:U_J\to M_{k\times(n-k)}(\mathbb{R})\equiv\mathbb{R}^{k(n-k)}$ by deleting the k columns corresp to J in the above representative ~~of~~ X of Y: $\varphi_J([Y])=X_{J'}$;

→ φ_J is well defined, is a homeomorphism onto $\mathbb{R}^{k(n-k)}$; $G(k,n)$ is covered by U_J, $J\subseteq(1,2,\cdots,n)$ $|J|=k$ & $\{(U_J,\varphi_J)\}$ is a C^∞-atlas on $G(k,n)$.

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