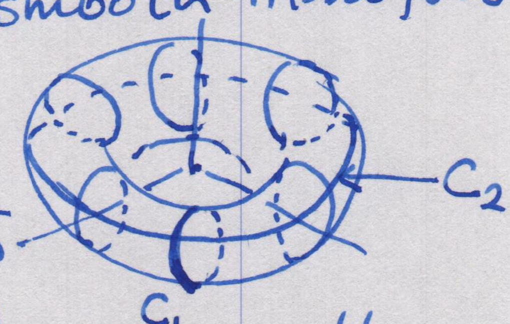
Lecture-7

<u>Examples (contd)</u> :- • Let M, N be C^r-manifolds of dimensions m & n respectively. If $\mathcal{M}$ & $\mathcal{N}$ be atlasses on M, N resp. Then $M \times N$ with the product topology (?) and the atlas $\{(U \times V, \varphi_U \times \psi_V) \mid (U, \varphi_U) \in \mathcal{M}, (V, \psi_V) \in \mathcal{N}\}$ gives $M \times N$ a C^r-mfld structure, $\dim M \times N = m+n$.

• The above example gives a smooth manifold structure on $T^2 := S^1 \times S^1$, where the first factor can be taken as $S^1 \equiv C_1$ and the second $S^1 \equiv C_2$, so T^2 is obtained by rotating C_1 about the Z-axis. More generally $T^n = S^1 \times \cdots \times S^1$ (n factors) is a smooth manifold.



<u>Exercise</u> :- Prove that the Grassmann manifold $G(k,n)$ as defined in Lecture-6, is compact;

[• $O(n, \mathbb{R})$ is compact & $O(n,\mathbb{R}) \xrightarrow{\phi} G(k,n)$

$\phi([c_1 \cdots c_n]) \longmapsto \langle c_1, c_2, \cdots, c_k \rangle$, c_i columns of A

$[c_1 \cdots c_n] = A \in O(n, \mathbb{R})$, $\langle c_1, \ldots, c_k \rangle$ = subspace of $\mathbb{R}^n$ spanned by $c_1, \ldots, c_k$

is continuous & surjective.]

<u>Differential structure on a manifold</u> :- Given a manifold M with two C^r-atlasses $\mathcal{A}_1$ & $\mathcal{A}_2$, we wish to ensure the notion of smoothness of a function on M is independent of the choice

of $\mathcal{A}_i$. To be precise say $f : M \to \mathbb{R}^m$ and we wish to discuss smoothness of f at $p \in M$. Let $(U, \varphi) \in \mathcal{A}_1$ be a chart, $p \in U$, $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$. We would like to call f smooth if $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is smooth, for <u>all</u> $(U, \varphi) \in \mathcal{A}_1$ with $p \in U$. It may well happen that f is smooth with respect to $\mathcal{A}_1$, but not wrt $\mathcal{A}_2$ (example?). To remove this ambiguity, we <u>define</u> two C^r-atlasses $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ & $\mathcal{B} = \{(V_\beta, \psi_\beta)\}$ to be <u>compatible</u> if for $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$ & $(V_\beta, \psi_\beta) \in \mathcal{B}$, $U_\alpha \cap V_\beta \neq \emptyset$, then

$$\varphi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \to \varphi_\alpha(U_\alpha \cap V_\beta) \text{ is } C^r.$$

- If $\mathcal{A}$ & $\mathcal{B}$ are compatible, then a function f on M is C^r wrt $\mathcal{A}$ iff f is C^r wrt $\mathcal{B}$.
- If $\mathcal{A}$, $\mathcal{B}$ be C^r-atlasses on M that are compatible, then the union $\mathcal{A} \cup \mathcal{B}$ is a C^r-atlas.
- Given a C^r-atlas $\mathcal{A}$, the atlas $\overline{\mathcal{A}} := \bigcup_{\mathcal{A}' \sim \mathcal{A}} \mathcal{A}'$, union over atlasses $\mathcal{A}'$ compatible with $\mathcal{A}$, is maximal, containing $\mathcal{A}$.

<u>Def^n</u>:- A C^r- maximal atlas on M is called

a differential structure on M. Hence we may think of a diff. structure on M as an equiv. class of atlasses with respect to compatibility (C^r). Note that while a topological manifold M has a unique C^0 structure, M may admit several different C^r-differential structures.

<u>Example</u>: $M = \mathbb{R}$; $\mathcal{A} = \{(\mathbb{R}, 1_{\mathbb{R}})\}$ is a C^∞-manifold, so is $M = \mathbb{R}$ with $\mathcal{B} = \{(\mathbb{R}, \varphi), \varphi(x) = x^3\}$

- $\mathcal{A}$ and $\mathcal{B}$ are <u>not</u> equivalent.

<u>Exercise</u>:- The atlasses $\{(\mathbb{R}, \varphi_i)\}_{i=0,1,2,\dots}$ are inequivalent smooth atlasses on $\mathbb{R}$; $\varphi_i : \mathbb{R} \to \mathbb{R}$, $\varphi_i(x) = x^{2i+1}$.

<u>Point set topology recalls</u>:- • We are aiming to do geometry on manifolds and the least we would want is put a metric on them, so we demand these be <u>Hausdorff</u> (metric spaces are such).

- A space X is <u>locally compact</u> if every point has an open nbd whose closure is compact. Manifolds are locally compact, as well as <u>locally path connected</u>, i.e. at every point, there is a <u>local basis</u> of path conn. open sets. ⇝

Exercise :- A connected topological manifold is path-connected.

Paracompact spaces :- These generalize compact spaces.

Let (X, τ) be a top. space and $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of X. An open covering $\{V_i\}_{i\in I}$ of X is called a refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$ if every set V_i is contained in some U_α.

A covering $\{V_i\}_{i\in I}$ is locally finite if every $x\in X$ has a nbd that intersects only finitely many V_i's. A top space X is paracompact if every open cover of X has a locally finite refinement. (see Munkres for details).

We need paracompactness property for manifolds because that ensures existence of 'partition of unity', a gadget that allows us to construct global objects from local data (we'll see this in detail, one example is the construction of a Riemannian metric).

Second countable space :- We call a top space (X, τ) second countable if it admits a countable basis.

- Let M be a top space that is Hausdorff and locally Euclidean (i.e. locally $\simeq \mathbb{R}^n$) and connected. Then the following are equivalent

7-5 (i) M is second countable. (ii) ~~M~~ M is paracompact.

More generally, dropping connected assumption on M, the following are equiv.

(i) M is 2nd countable. (ii) M is paracompact and has countably many conn. components.

Finally: (this does what we need/want!)

M (Hausdorff), locally compact and second countable then M is paracompact. (Munkres)

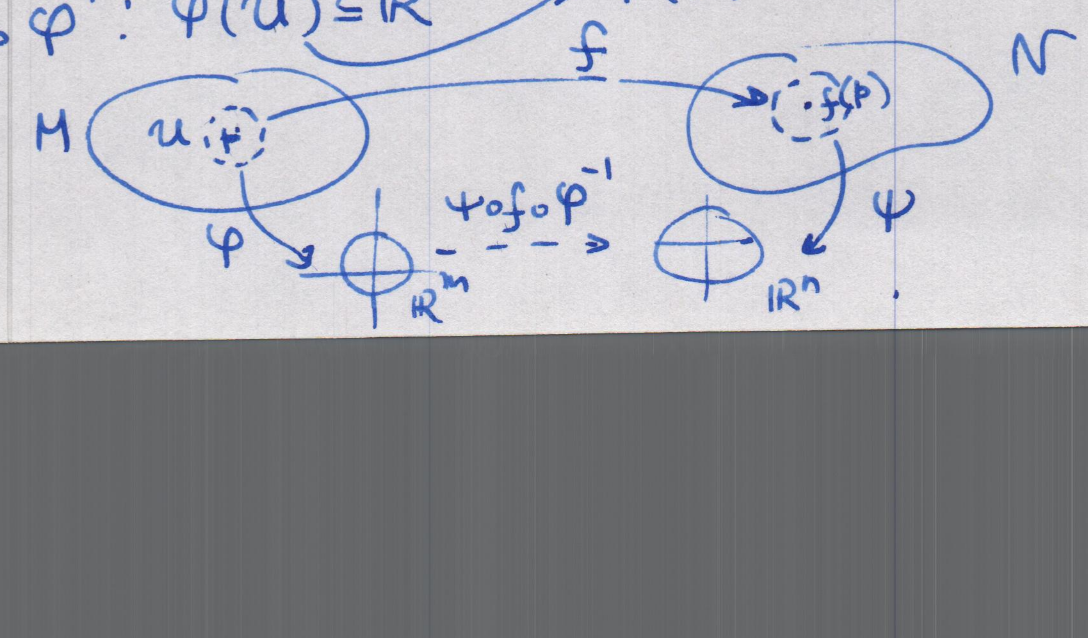
<u>Def^n</u>:- A <u>smooth manifold</u> M of dimension n is a Hausdorff topological space satisfying

- M is 2nd countable, paracompact together

with a <u>C^∞-differential = differentiable structure</u> (i.e. equiv. class of a C^∞-atlas).

<u>We'll work with this def^n now on</u>. We now define:

<u>Smooth maps between manifolds</u>:- Let M, N be smooth manifolds, $U \subseteq M$ open. A map $f: U \to N$ is <u>smooth</u> if $\forall p \in U$, $\exists$ <u>coordinate charts</u> $(\mathcal{U}, \varphi)$ around p and (V, ψ) around $f(p)$ such that $f(\mathcal{U}) \subseteq V$ and the map $\psi \circ f \circ \varphi^{-1} : \varphi(\mathcal{U}) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$ is smooth:



- $M \xrightarrow{f} N \xrightarrow{g} S$ smooth $\Rightarrow g\circ f : M \to N$ is smooth.
- M_1, M_2 smooth $\Rightarrow$ projections $M_1 \times M_2 \xrightarrow{\pi_i} M_i$ $i=1,2$, are smooth.

<u>Algebra of smooth functions</u> :- Let M be a smooth manifold. Let $C^\infty(M) = \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}$. For $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M)$, define $\alpha f + \beta g \in C^\infty(M)$ as usual and fg by $(fg)(x) = f(x)g(x)$. Then $fg \in C^\infty(M)$. Hence $C^\infty(M)$ is an $\mathbb{R}$-algebra with these operations

<u>More generally</u> for $U \subseteq M$ open, we have $C^\infty(U)$ as the set of smooth functions on U.

- $C^\infty(U)$ is an $\mathbb{R}$-algebra, containing constants.
- $V, U \subseteq M$ open, $V \subseteq U$ then $f \in C^\infty(U)$ $\Rightarrow f \in C^\infty(V)$.
- If $U = \bigcup_i U_i$, $U_i \subseteq M$ open and $f : U \to \mathbb{R}$ be such that $f|_{U_i} \in C^\infty(U_i)\ \forall\, i$, then $f \in C^\infty(U)$.
- $\exists$ an integer $m \geq 0$ such that $\forall\, p \in M$, $\exists$ an open set $U \ni p$ and functions (distinct) $x_1, \dots, x_m \in C^\infty(U)$ such that
  (i) the map $\varphi : q \mapsto (x_1(q), \dots, x_m(q))$ is a homeomorphism $U \simeq$ ~~$\mathbb{R}^m$~~ $\underset{\text{open}}{W \subseteq} \mathbb{R}^m$.
  (ii) $\forall$ open $V \subseteq U$ & $f : V \to \mathbb{R}$, $f \in C^\infty(V)$ $\Leftrightarrow f\circ\varphi^{-1} \in C^\infty(\varphi(V))$.

7-7. The proofs are left to you. The above is a very useful way to <u>define</u> a smooth manifold.

<u>Example</u>: <u>Lie groups</u> :- A Lie group is a group G with a smooth $\equiv$ differentiable mfld structure and we <u>demand</u> that the product $\mu: G \times G \to G$, $\mu(x,y) = xy$ and the inverse $i: G \to G$, $x \mapsto x^{-1}$ are both smooth. (equiv. $(x,y) \mapsto xy^{-1}$ is smooth).

- $(\mathbb{R}^n, +)$, $(\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \cdot)$, $(S^1, \cdot)$ are Lie groups.
- $GL_n(\mathbb{R})$, $O(n,\mathbb{R})$, $U(n)$, $SO(n)$, $SU(n)$, $SL(n,\mathbb{R})$ are Lie groups.

<u>Exercise</u>: G be a Lie group and $a, b, g \in G$. Let $R_a, L_b, A_g : G \to G$ be the maps $R_a(x) = xa$, $L_b(x) = bx$, $A_g(x) = gxg^{-1}$. These are smooth maps, hence (?) are diffeomorphisms. $L_a \equiv$ left translation by a, $R_a \equiv$ right transl^n, $A_g \equiv$ Conjugation by g.

Having defined the notion of smooth = diff. functions between manifolds, we would like to discuss derivatives of such. →

7-8 Going by the case of (open subsets of) $\mathbb{R}^n$, the derivative at a point should be a linear map, but the domain and codomain are not clear at this stage! We need to first define

<u>Tangent space</u> :- Philosophically speaking, since a manifold does not come with an embedding into a Euclidean space, we cannot use the techniques we used in case of (open subsets / curves, surfaces) $\mathbb{R}^n$. This will be defined 'intrinsically', using the diff. structure at hand. In this, we'll see several 'avatars' of the tangent space, all equally useful and we'll have natural isomorphisms between them, allowing us to pass from one description to the other. First let us discuss these ideas for open subsets of $\mathbb{R}^n$.

<u>Directional derivative</u>: Let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ and $v \in \mathbb{R}^n$. The <u>directional derivative</u> <u>of f at p in the direction of v</u> is the <u>limit</u>

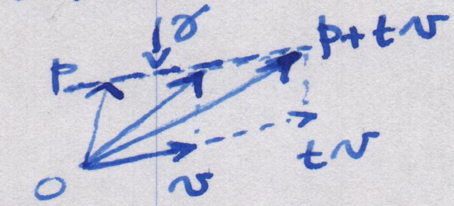
$$D_v f(p) := \lim_{t \to 0} \frac{1}{t} \left[f(p+tv) - f(p) \right],$$

if it exists. Let $g(t) := f(p+tv) \in \mathbb{R}^m$.

7-9 Then for $I \ni 0$, an open interval $\subseteq \mathbb{R}$, we have

(*) $$D_v f(p) = \frac{d}{dt} g(t)\Big|_{t=0} = Df(p)(v).$$

If $\gamma(t) = p + tv$, then γ is a linear curve with $\gamma(0) = p$ & $\dot{\gamma}(0) = v$.



The point here is that for a small ball $\ni p$ and small values of t, $\gamma(t)$ lies in the ball. So, to evaluate the limit (*) above, we restrict f to the curve $\gamma(t)$. We could have used any curve for this:

<u>Lemma</u>:- Let $\sigma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a smooth curve with $\sigma(t) \in U$ $\forall t \in (-\epsilon, \epsilon)$; $\sigma(0) = p$ and $\dot{\sigma}(0) = v$, then

$$D_v f(p) = \frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0}.$$

<u>Proof</u>:- By Chain rule we have

$$\frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0} = Dg(p)(v)$$
$$= Df(p)(\dot{\sigma}(0))$$
$$= Df(p)(v) = D_v f(p).$$

$\boxtimes$

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$\rightsquigarrow$

Hence we can think of the domain of $Df(p)$ as the set of tangent vectors to curves passing through p and lying in U. We have shown that $D_v f(p)$ is independent of the choice of a curve γ in U with $\gamma(0) = p$; $\dot{\gamma}(0) = v$.

Let $U \subseteq \mathbb{R}^n$ be open, $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a curve lying in U, $\gamma(0) = p$. Let $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$. Then the tangent to γ at p is $\dot{\gamma}(0) = (\dot{\gamma}_1(0), \dots, \dot{\gamma}_n(0))$. Let $v \in \mathbb{R}^n$ be any vector. Since U is open, & $p \in U$, $\exists\ \delta > 0$ s.t. $B(p, \delta) \subseteq U$ and hence the linear curve $\gamma = \gamma_{p,v}(t) = p + tv$ as above, lies in U & has $\gamma(0) = p$, $\dot{\gamma}(0) = v$.

Hence the set $T_p(U)$ of all tangent vectors at p can be identified with $\mathbb{R}^n$.

We may regard two curves σ, γ through p, lying in U, as <u>equivalent</u> if $\dot{\sigma}(0) = \dot{\gamma}(0)$, i.e. σ and γ have the same tangent vector at p.

<u>Exercise</u>: (1) Show that this equiv is indep. of choice of $U \ni p$, i.e. $T_p(U) = T_p(\Omega)$ for any open $\Omega \ni p$.

(2) The above indeed is an equiv. rel^n.