More generally, when A is a commutative ring and M an A-module, we define $\delta : A \to M$ as above. The directional derivative operators, by the lemma, are precisely the 'point derivations' on $C^\infty(p)$, i.e. those satisfying the product formula $\delta(fg) = f(p)\delta(g) + \delta(f)g(p)$. Hence we have

$$T_p(\mathbb{R}^n) \equiv T_p(U) = \left\{ [\gamma] \;\middle|\; \begin{array}{l} \gamma \subseteq U \text{ smooth}, \\ \gamma(0) = p \end{array} \right\} \cancel{\dot{\gamma}(0)}$$

$$= \left\{ \delta \mid \delta \text{ a point derivation on } C^\infty(p) \right\}.$$

• **Remarks**:- (i) In finding/solving for 'v' for a given point derivation δ, we need in $(*)$ that f be smooth; if f were only C^1 e.g. then the g_i's in $(*)$ may be only continuous and then δ cannot be applied.

(ii) Given $f : U \to \mathbb{R}^m$, $p \in U$, $f = (f_1, \dots, f_m)$, $f_i = x_i \circ f$, we have

$Df(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. We may now think of $\mathbb{R}^n = T_p(U)$, $\mathbb{R}^m = T_{f(p)}(\mathbb{R}^m)$

and identify $Df(p)(v) = (Df_1(p)(v), \ldots, D(f_m)(p)(v))^t$ as a column vector and hence $Df(p)(v)$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\big|_p & \cdots & \frac{\partial f_1}{\partial x_n}\big|_p \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}\big|_p & \cdots & \frac{\partial f_m}{\partial x_n}\big|_p \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

<u>Tangent space to a manifold</u> :- Let now M be a smooth manifold, $p \in M$. Let

$C^\infty(p)$ = set of smooth $\mathbb{R}$-valued functions defined in a nbd of p; i.e. $f : U_f \to \mathbb{R}$ is smooth on the open set $U_f \subseteq M$.

<u>Def</u> :- A <u>tangent vector</u> v <u>at $p \in M$</u> is a point derivation of $C^\infty(p)$, i.e. $v : C^\infty(p) \to \mathbb{R}$ is $\mathbb{R}$-linear and

$$v(fg) = f(p)\, v(g) + v(f)\, g(p), \quad f, g \in C^\infty(p).$$

We denote the $\mathbb{R}$-vector space of all tangent vectors to M at p by $T_p(M)$.

<u>Some elements of $T_p(M)$</u> :- Let (U, φ) be a coordinate chart around p and $\varphi(p) = 0$ (?). Let x_i $(1 \le i \le n)$ denote the coordinate functions on U and u_i be the coordinates

in $\mathbb{R}^n$. Then, for $f \in C^\infty(p)$, define

$$\frac{\partial}{\partial x_i}\Big|_p (f) := \frac{\partial}{\partial u_i}(f\circ\varphi^{-1})(0)$$
$$= \frac{\partial}{\partial u_i}\Big|_0 (f\circ\varphi^{-1}).$$

Then $\frac{\partial}{\partial x_i}\Big|_p \in T_p(M)$ (check this, easy!)

- When $f = x_i$, $\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial}{\partial u_i}(\underbrace{x_i\circ\varphi^{-1}}_{=u_i})(0)$
$$= \frac{\partial}{\partial u_i}(u_i)(0) = 1.$$

<u>Derivatives</u> of <u>curves on M</u> :- Let $\gamma : (-\epsilon,\epsilon)\to M$ be a smooth curve through p, i.e. $\gamma(0)=p$.
Define $\dot\gamma(0)$ by the point derivation

$$\dot\gamma(0)(f) := \frac{d}{dt}(f\circ\gamma)(0) =: \frac{d}{dt}\Big|_{t=0}(f\circ\gamma).$$

Then

- $\dot\gamma(0)(f+g) = \dot\gamma(0)(f) + \dot\gamma(0)(g)$
- $\dot\gamma(0)(fg) = \frac{d}{dt}((fg)\circ\gamma)(0)$
$$= f(p)\,(\dot\gamma(0)(g)) + (\dot\gamma(0)(f))\,g(p)$$

as is easily checked (see Kumaresan).
Hence $\dot\gamma(0) \in T_p(M)$. We have $\rightsquigarrow$

Theorem:- Let M be a smooth manifold, $p \in M$ and (U, φ) be a chart around p, x_i the local coordinates on U, $1 \leq i \leq m = \dim M$.

- Let $v \in T_p(M)$. Then $v = \sum_i v(x_i) \frac{\partial}{\partial x_i}\Big|_p$
- $\left\{ \frac{\partial}{\partial x_i}\Big|_p, 1 \leq i \leq m \right\}$ are lin. indep/$\mathbb{R}$.
- $\dim T_p M = m = \dim M$.

Proof:- Assume wlg that $\varphi(p) = 0$. Then $\varphi(U) \subseteq \mathbb{R}^m$ is an open set $\ni 0$ and $F := f \circ \varphi^{-1} \in C^\infty(0)$.
Then the Taylor formula for F gives

$$(\#) \qquad F(x) = F(0) + \sum_i u_i G_i(x), \text{ where}$$

G_i are smooth around 0 & $G_i(0) = \frac{\partial}{\partial u_i} F(0)$.
Composing $(\#)$ by φ yields

$$(*) \qquad f(x) = f(p) + \sum x_i g_i(x);$$

$$x_i = u_i \circ \varphi, \; g_i = G_i \circ \varphi.$$

Hence $$v(f) = v(f(p)) + \sum_i \left[v(x_i) g_i(p) + x_i(p) v(g_i) \right].$$

We have $g_i(p)$

$$= G_i \circ \varphi(p) = G_i(0) = \frac{\partial}{\partial u_i}\Big|_0 (F) = \frac{\partial}{\partial u_i}\Big|_0 (f \circ \varphi^{-1}).$$

We defined $\frac{\partial}{\partial x_i}\Big|_p (f) := \frac{\partial}{\partial u_i}\Big|_0 (f \circ \varphi^{-1})$. Hence
$g_i(p) = \frac{\partial}{\partial x_i}\Big|_p (f)$.

$\longrightarrow$

8-9. But $v(f(p)) = 0 = x_i(p)$ as $\varphi(p) = 0$. Hence

$$v = \sum_i v(x_i) \frac{\partial}{\partial x_i}\Big|_p.$$

[discuss the case of φ when $\varphi(p) \neq 0$]. ⊠

<u>Linear independence of $\frac{\partial}{\partial x_i}\big|_p$</u>: We have already computed $\frac{\partial}{\partial x_i}\big|_p (x_i) = 1$. We have

$$\frac{\partial}{\partial x_i}\Big|_p (x_j) = \frac{\partial}{\partial u_i}(x_j \circ \varphi^{-1})(0) = \frac{\partial}{\partial u_i}(u_j)(0)$$

$= \delta_{ij}$. Hence $\left\{ \frac{\partial}{\partial x_i}\big|_p , 1 \leq i \leq m \right\}$ form a basis of $T_p(M)$ and thus $\dim_{\mathbb{R}} T_p(M) = m$.

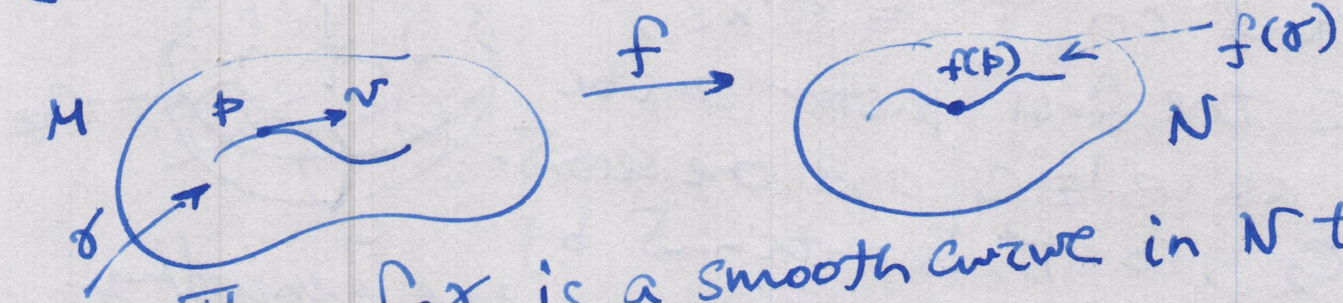
*<u>Lemma</u>: For any $v \in T_p(M)$, $\exists$ a smooth curve γ on M, $\gamma(0) = p$ & $\dot{\gamma}(0) = v$.

<u>Proof</u>: Class discussion exercise. ⊠

This Lemma allows us to identify $T_p(M)$ with the set of equivalence classes of smooth curves γ on M with $\gamma(0) = p$; under the equiv. $\gamma \sim \sigma$ if $\gamma(0) = \sigma(0) = p$ & $\dot{\gamma}(0) = \dot{\sigma}(0)$ as elements of $\mathrm{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$ = point derivations on $C^\infty(p)$.

8-10

Derivatives of smooth maps:- Let M, N be smooth manifolds of dim m, n resp, $f: M \to N$ be a smooth map. In analogy with the case of $M = U \underset{\text{open}}{\subseteq} \mathbb{R}^m$, $N = \mathbb{R}^n$, we would like the derivative of f at $p \in M$ to be a linear map $Df(p): T_pM \to T_{f(p)}N$. We use the earlier Lemma for this. Let $v \in T_pM$. Then $\exists$ a smooth curve γ on M, $\gamma(0) = p$, $\dot{\gamma}(0) = v$.



Then $f \circ \gamma$ is a smooth curve in N through $f(p)$. We can therefore define

$$Df(p)(v) := (f \circ \gamma)'(0).$$

Since $f \circ \gamma$ is a smooth curve through $f(p)$ on N, $(f \circ \gamma)'(0) \in T_{f(p)}(N)$.

- $Df(p)(v)$ is well defined.
- $Df(p): T_pM \to T_{f(p)}N$ is a linear map.
- $Df(p)(v)(g) = v(g \circ f)$, $g \in C^\infty(f(p))$.

These are left to you as exercises.