# Lecture-8

In the last lecture we started a discussion on tangent spaces and saw that for $U \subseteq \mathbb{R}^n$ open and $p \in U$, $T_p(U) = \mathbb{R}^n$, where we set $T_p(U)$ = set of equiv. classes of smooth curves in $U$ under the equiv. rel$^n$ $\sigma \sim \gamma$ if $\sigma(0) = \gamma(0) = p$, $\dot\sigma(0) = \dot\gamma(0)$. [curves are 1-dim'l mfds].

If we write $[\gamma]$ for the equiv. class of $\gamma$ for $\gamma$ as above, then $T_p(U) = \{[\gamma] \mid \gamma \subseteq U\}$.

## Tangent vectors as directional derivative operators

We work with $U \subseteq \mathbb{R}^n$ as above. Let $v \in \mathbb{R}^n$ $(= T_p(U), p \in U)$ and $f \in C^\infty(U)$. Then, for $x \in U$, we have the directional derivative $D_v f(x)$, of $f$ at $x$, in the direction of $v$;

$$D_v f(x) = \lim_{t \to 0} [f(x+tv) - f(x)]$$

$$= \langle \operatorname{grad} f(x), v \rangle.$$

For $p \in U$, $D_v f(p) = \langle \nabla f(p), v \rangle = \sum_i v_i \frac{\partial f}{\partial x_i}\Big|_p$

$$=: \sum_i v_i \frac{\partial}{\partial x_i}\Big|_p (f) \longrightarrow$$

where, in the last expression we wish to think of $f \longmapsto \sum_i v_i \frac{\partial f}{\partial x_i}\Big|_p$ as an operator on $C^\infty(U)$. We have noticed before that to compute $D_v f(p)$, we use <u>any</u> curve $\gamma$ in $U$, $\gamma(0)=p$ and $\dot\gamma(0)=v$, without affecting the value of $\lim_{t\to 0}\frac{1}{t}(f(p+tv)-f(p))$

$= \frac{d}{dt}(f\circ\gamma(t))\Big|_{t=0}$ (see Lemma, Lecture-7).

We can <u>therefore</u> associate <u>the operator</u> $f \mapsto D_v(f)(p) =: D_{p,v}(f)$ to the equiv. class $[\gamma]$, $\gamma(0)=p$, $\dot\gamma(0)=v$.

- $D_{p,v}(f) = D_{p,w}(f)$ $\forall f \in C^\infty(U)$, $U \ni p$; $v=(v_1,\dots,v_n)$, $w=(w_1,\dots,w_n)$

$\Rightarrow D_{p,v}(x_i) = v_i = D_{p,w}(x_i) = w_i$, $1\le i\le n$.

Hence $v=w$. So the map $\mathbb{R}^n \to \{D_{p,v} \mid v\in\mathbb{R}^n\}$
$v \longmapsto D_{p,v}$

is injective.

- The set $\{D_{p,v} \mid v\in\mathbb{R}^n\}$ is a vect sp/$\mathbb{R}$:

$(\alpha D_{p,v} + \beta D_{p,w})(f) := \alpha . D_{p,v}(f) + \beta . D_{p,w}(f)$,

$\alpha,\beta\in\mathbb{R}$. Hence $v \mapsto D_{p,v}$ is an isom. of vect

$\mathbb{R}$-vector spaces. We note that the ops $D_{p,v}$ have the following properties:

(i) $D_{p,v}(f) = D_{p,v}(g)$ for $f, g$ smooth in a nbd $\ni p$ (possibly different nbds) and $f = g$ on a nbd $\ni p$.

• This is easy, since the nbds of def$^n$/ smoothness for $f$ & $g$ intersect $\neq \phi$, say in $W$, then $D_{p,v}(f)$ & $D_{p,v}(g)$ can be computed, e.g. using curve $\subseteq W$.

(ii) $D_{p,v}(\alpha f + \beta g) = \alpha D_{p,v}(f) + \beta D_{p,v}(g)$, $\alpha, \beta \in \mathbb{R}$.

(iii) $D_{p,v}(fg) = f(p) D_{p,v}(g) + D_{p,v}(f) g(p)$

$$= \sum_i v_i \frac{\partial}{\partial x_i}(fg)\Big|_p = \sum_i \left[ v_i \left(\frac{\partial f}{\partial x_i}\Big|_p\right) g(p) + f(p) \frac{\partial}{\partial x_i}(g)\Big|_p \right] \text{etc.}$$

<u>Lemma</u> (Characterization of directional derivative operators): Let $C^\infty(p)$ = $\mathbb{R}$-vector space of $\mathbb{R}$-valued functions, smooth in a nbd of $p$ (the nbd may depend on a function). Then for an $\mathbb{R}$-linear map $\delta : C^\infty(p) \to \mathbb{R}$ satisfying (iii) above: $\delta(fg) = f(p)\delta(g) + \delta(f) g(p)$ $\rightarrow$

$\forall f, g \in C^\infty(p)$ (this is an $\mathbb{R}$-algebra (?)), then $\exists!$ $v \in \mathbb{R}^n$ such that $\delta = D_{p,v}$.

Proof:- We'll sketch this (see Pati/Kumaresan).

- For constant functions $c \in C^\infty(p)$, $\delta(c) = 0$ (follows from the product formula).
- May assume $p = 0 =$ origin in $\mathbb{R}^n$ (?).
- Taylor formula for $f$ around $0 \Rightarrow$

(*) $f(x) = f(0) + \sum_i x_i g_i(x)$, $x = (x_1, \dots, x_n)$

in a small ball around $0$ & $g_i$'s are smooth on this ball, $g_i(0) = \frac{\partial f}{\partial x_i}\Big|_0$.

- by (*), $\delta(f) = \delta\left(f(0) + \sum_i x_i g_i(x)\right)$

$$= \sum_i \delta(x_i) g_i(0) \quad \text{[given by product formula for } \delta]$$

$$= \left(\sum \delta(x_i) \frac{\partial}{\partial x_i}\Big|_0\right)(f).$$

- $v := (\delta(x_1), \dots, \delta(x_n)) \Rightarrow \delta = D_{p=0, v}$. ☒

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Point derivations:- Given an $\mathbb{R}$-algebra $A$, a derivation, with values in $\mathbb{R}$, is an $\mathbb{R}$-linear map $\delta : A \to \mathbb{R}$ such that

$$\delta(ab) = \delta(a) b + a \delta(b), \quad a, b \in A.$$