Lecture-9

Remarks :- In the last lecture we developed the notion of tangent vector to a smooth manifold at a point and the notion of the derivative of a smooth map between two such manifolds. In this, we dealt with the $\mathbb{R}$-algebra $C^\infty(p)$ of smooth $\mathbb{R}$-valued functions, smooth in a nbd of $p \in M$. We may think of $C^\infty_p(p) = \{(f, U_f) \mid f \text{ smooth on } U_f, \text{ open nbd of } p\}$. The rel^n

$(f, U_f) \sim (g, U_g)$ if $f = g$ on $U_f \cap U_g$ is an equiv. rel^n on $C^\infty(p)$. The class $[(f, U_f)]$ is called germ of f at p. The set of germs of smooth functions $= \mathcal{O}_p$ (M understood) is an $\mathbb{R}$-algebra: $[(f, U_f)] + [(g, U_g)] := [(f+g, U_f \cap U_g)]$, $[(f, U_f)] \cdot [(g, U_g)] = [(fg, U_f \cap U_g)]$ etc.

- The evaluation at p map $\mathcal{O}_p \xrightarrow{e_p} \mathbb{R}$, $[(f, U_f)] \mapsto f(p)$ is surjective $\mathbb{R}$-algebra map (?). Let $\ker e_p =: \mathfrak{m}_p$. Then $\mathfrak{m}_p$ is a max ideal.
- Exercise: $\mathcal{O}_p$ is a local ring, i.e. has a unique max ideal.

We have the induced isom. $\mathcal{O}_p / \mathfrak{m}_p \simeq \mathbb{R}$.
mapping $[(f, U_f)] + \mathfrak{m}_p \longmapsto f(p)$.
We have therefore an $\mathcal{O}_p$-module structure on $\mathbb{R}$; $[(f, U_f)] \cdot r := f(p)r$ (check).
We therefore can view a 'point derivation' on $C^\infty(p)$ as we defined, as a derivation $\delta : \mathcal{O}_p \longrightarrow \mathbb{R}$, $\mathbb{R}$ treated as $\mathcal{O}_p$-module via the evaluation action. Then

$$\delta([f][g]) = [f] \cdot \delta([g]) + \delta([f]) \cdot [g]$$
$$= f(p)\,\delta([g]) + \delta([f])\, g(p).$$

- Let M be a smooth manifold of dim m. Given $p \in M$ and $v \in \mathbb{R}^m$, how do we interpret v as a tangent vector $\equiv$ derivation, at p? For this, fix a chart (U, φ) around p, $\{u_1, \cdots, u_m\}$ be the Euclidean (usual) coordinates on $\varphi(U)$, $\varphi(p) = 0 \in U$ and $x_i = u_i \circ \varphi$, $1 \le i \le m$. Then v corresponds to the derivation $\sum_i v_i \frac{\partial}{\partial x_i}\big|_p$ on $\mathcal{O}_p$, $v = (v_1, \cdots, v_m)$.

- $p \in M \xrightarrow{f} N \hookleftarrow\supseteq V \xrightarrow{g} \mathbb{R}$, $f(p) \in V \underset{\text{open}}{\subseteq} N$. Then

$$Df(p)(v) \in T_{f(p)}N \;\; \& \;\; Df(p)(v)(g) = v(g \circ f)$$

$\forall\, g \in C^\infty_{f(p)}(N)$ or $[g] \in \mathcal{O}_{f(p)}$ $\rightarrow$

Matrix representation of $Df(p)$:- Let $q = f(p)$, (U, φ) be a chart around p, $\varphi(p) = 0$ & (V, ψ) a chart around q, $\psi(q) = 0$. Let $\{x_i\}$ & $\{y_j\}$ be the coordinates on U & V resp. so

$x_i(z) = u_i(\varphi(z))$ $\forall$ $z \in U$, $y_j(z) = u_i(\psi(z))$ $\forall$ $z \in V$, $\{u_i\}_i$ being the Euclidean coords.

Then, as we proved, $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_i$ and $\left\{ \frac{\partial}{\partial y_j}\Big|_q \right\}_j$ are bases for $T_p(M)$ and $T_q(N)$ resp. $Df(p) : T_p M \to T_q N$ is a linear map, we wish to write its matrix wrt the above bases. Write $v = \frac{\partial}{\partial x_i}\Big|_p$, $\omega = Df(p)(v)$.

Then
$$\omega = \sum_j \omega(y_j) \frac{\partial}{\partial y_j}\Big|_q = \sum_j [Df(p)(v)](y_j) \frac{\partial}{\partial y_j}\Big|_q$$

$$= \sum_j v(y_j \circ f) \frac{\partial}{\partial y_j}\Big|_q = \sum_j \frac{\partial}{\partial x_i}\Big|_p (y_j \circ f) \frac{\partial}{\partial y_j}\Big|_q$$

Hence the matrix for $Df(p)$ is $\left(\frac{\partial}{\partial x_i}\Big|_p (y_j \circ f) \right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

$n =: \dim N$. Which is the usual Jacobian matrix in the Euclidean set up. $\boxtimes$

- Using the formula $D(f)(p)(g) = v(g \circ f)$, it is clear that $D^{\omega}(f)(p)(\omega) : T_p M \to T_{f(p)} N$ is linear.

- For $I: M \to M$, the identity map, $D(I)(p)$ $: T_pM \to T_pM$ is identity. Since, for $v \in T_pM$ & $g \in C^\infty_p(p)$, ~~v~~ $D(I)(p)(v)(g) = v(g \circ I)$
$= v(g)$.

Hence $D(I)(p)(v) = v \;\forall\; v \in T_p(M)$.

- <u>Chain rule</u>: $M \xrightarrow{f} N \xrightarrow{g} W$ smooth, then $D(g\circ f)(p) = Dg(f(p)) \circ Df(p)$.

We have, for $\varphi \in$ ~~$C^\infty(p)$~~ $C^\infty(g\circ f(p))$, $v \in T_pM$

$$[D(g\circ f)(p)(v)](\varphi) = v(\varphi \circ (g\circ f))$$
$$= v((\varphi\circ g)\circ f). \text{ Now}$$

$\varphi \circ g \in C^\infty(f(p))$. Hence $\Big\} = Df(p)(v)(\varphi\circ g)$

$$= D(g)(f(p))\big(Df(p)(v)\big)(\varphi). \quad \boxtimes$$

<u>Corollary</u>:- Let $\varphi: M \to N$ be a diffeomorphism with inverse $\psi = \varphi^{-1}: N \to M$. For $p \in M$,

$D\varphi(p): T_pM \longrightarrow T_{\varphi(p)}N$ is bijective and

$$(D\varphi(p))^{-1} = D\psi(\varphi(p)).$$

<u>Proof</u>: Observe: $D\psi(\varphi(p)) = D(\psi\circ\varphi)(p) = D(I)(p)$
$= Id_{T_pM}$

$$D\varphi(p) \circ D\psi(\varphi(p)) = D(\varphi\circ\psi)(\varphi(p)) = D(Id_N)(\varphi(p))$$
$$= Id_{T_{\varphi(p)}N}. \quad \boxtimes$$

Exercise :- Let M_1, M_2 be smooth, $\pi_i : M_1 \times M_2 \to M_i$ be the projections. These are differentiable. Compute $D\pi_i$, $i=1,2$.

Examples :- 1. Let $M = (-\epsilon, \epsilon)$, $\epsilon > 0$ with the usual manifold structure (as open $\subseteq \mathbb{R}$) and N be a smooth manifold. Let $\gamma : M \to N$ be smooth. Then, for $t \in M$, $D\gamma(t)\left(\frac{d}{du}\Big|_t\right) \in T_{\gamma(t)}N$. We have, for $f \in C^\infty(\gamma(t))$,

$$D\gamma(t)\left(\frac{d}{du}\Big|_t\right)(f) = \frac{d}{du}\Big|_t \left(f \circ \gamma(t)\right).$$

For $p = \gamma(0)$, let (U, φ) be a coordinate chart in N with $\varphi(p) = 0 \in \mathbb{R}^n$; then, by def^n of $\dot{\gamma}(0)$ we have, in terms of the coordinates $x_1, \ldots, x_n$ on U, $\dot{\gamma}(0) = \sum \frac{d\gamma_i}{du}\Big|_0 \frac{\partial}{\partial x_i}\Big|_p$, where $\gamma_i = x_i \circ \gamma$ in U.

2. In the above we used the chart $u : t \mapsto t$ on $(-\epsilon, \epsilon) \subseteq \mathbb{R}$ and $T_0 M = \mathbb{R} \cdot \frac{d}{du}\Big|_0$. If $f \in C^\infty(N)$ and $v \in T_pN$, then $Df(p)(v) \in T_{f(p)}(\mathbb{R}) = \mathbb{R} \cdot \frac{d}{du}\Big|_{f(p)}$ and $Df(p)(v) = \alpha(v) \cdot \frac{d}{du}\Big|_{f(p)}$ where $\alpha(v) = v(f) \in \mathbb{R}$. To see this, we compute:

$$Df(p)(v) = \left[Df(p)(v)\right](u) \cdot \frac{d}{du}\Big|_{f(p)}$$
$$= v(u \circ f) \cdot \frac{d}{du}\Big|_{f(p)} = v(f) \cdot \frac{d}{du}\Big|_{f(p)}.$$

Cotangent space :- Let M be a smooth manifold, $p \in M$. Then $(T_pM)^* = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$ is called the cotangent space to M at p and elements of T_pM^* are called cotangent vectors.

Let $f \in C^\infty(M)$, so $f : M \to \mathbb{R}$ is smooth. Hence $Df(p) : T_pM \to T_{f(p)}\mathbb{R} \equiv \mathbb{R}$ is given by $Df(p)(v) = v(f) \frac{d}{du}\big|_{f(p)}$ as we discussed. We can therefore think of $Df(p)$ as a linear functional on T_pM; $v \mapsto v(f)$. We'll denote this by $df(p)$, i.e. $df(p)(v) = v(f)$.

If (U, φ) be a chart around p with coords $x_1, \dots, x_m$; we get $dx_i(p) \in T_pM^*$. Let us compute $dx_i(p)\left(\frac{\partial}{\partial x_j}\big|_p\right) = \frac{\partial}{\partial x_j}\big|_p (x_i|_p)$

$$= \delta_{ij}.$$

Hence $dx_1(p), \dots, dx_m(p)$ are lin indep $/\mathbb{R}$ and form a basis of T_pM^*. By the above, $\{dx_1(p), \dots, dx_m(p)\}$ is the dual basis corresp to $\left\{\frac{\partial}{\partial x_1}\big|_p, \dots, \frac{\partial}{\partial x_m}\big|_p\right\}$, $dx_i(p) = \left(\frac{\partial}{\partial x_i}\big|_p\right)^*$. $\rightsquigarrow$

- For $f \in C^\infty(M)$, we can think of $p \mapsto df(p)$ as the map $df : M \to \bigcup_p T_pM^*$ which has $df(p) \in T_pM^*$. We can write a <u>local expression</u> for df:

<u>Lemma</u>:- Let ($p \in M$ and) $x_1, \dots, x_m$ be local coordinates (around p) on $U \ni p$, then,

$$df(x) = \frac{\partial f}{\partial x_1}(x)\, dx_1 + \cdots + \frac{\partial f}{\partial x_m}(x)\, dx_m.$$

<u>Proof</u>:- For $p \in U$ and $v \in T_pM$ we have

$$df|_p(v) = df|_p\left(v_1 \frac{\partial f}{\partial x_1}\Big|_p + \cdots + v_m \frac{\partial f}{\partial x_m}\Big|_p\right).$$

On the other hand, $dx_i|_p\left(v_1 \frac{\partial}{\partial x_1}\Big|_p + \cdots + v_m \frac{\partial}{\partial x_m}\Big|_p\right)$

$= v_i$ by the calcul^n $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$.

Hence $df|_p(v) = \frac{\partial f}{\partial x_1}\Big|_p dx_1|_p(v) + \cdots + \frac{\partial f}{\partial x_m}\Big|_p (v)\, dx_m|_p$.

$\boxtimes$

$$\therefore\ df|_p = \sum_i \frac{\partial f}{\partial x_i}\Big|_p dx_i|_p.$$

<u>Remarks</u>:- 1. For $f \in C^\infty(M)$, df as above is what is called a (smooth) differential 1-form (linear functionals are called <u>linear</u> forms) on M. We'll study these in general soon.

$\rightsquigarrow$

2. The spaces $\bigcup_p T_p M$ and $\bigcup_p T_p M^*$ carry natural differential structures and hence are smooth manifolds, denoted by $T(M)$ & $T(M)^*$, called the tangent and cotangent bundle of M resp. We'll discuss these.

$\boxed{\text{Submanifolds}}$:- Given a manifold M and $S \subseteq M$ we would like to define S as a smooth submanifold if S is a manifold in its own right, with some compatibility conditions. Before we proceed, we need to study maps $f: M \to N$ with conditions on Df.

Immersion, submersion:- Let $f: M \to N$ be smooth and $p \in M$. We call f an immersion at p if $Df(p)$ is injective, submersion at p if $Df(p)$ is surjective and f is a local diffeomorphism at p if $Df(p)$ is bijective: $T_p M \to T_{f(p)} N$.

Call f an immersion if f is an immersion at all $p \in M$, similarly define f to be a submersion.

Example (Canonical immersion / submersion):

- For $m \leq n$, $\iota : \mathbb{R}^m \to \mathbb{R}^n$ be the map $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$. Clearly this is an immersion, called the canonical immersion.

If $m \geq n$, the map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ given by $(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n)$ is clearly a submersion, called the canonical submersion.

<u>A non-example</u> :- A smooth injective map may fail to be an immersion, e.g. $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x^3$ is not one (why?).

<u>Theorem</u> :- 1. Let $f : M \to N$ be an immersion on a neighbourhood of $p \in M$. Then $\exists$ coord charts $(U, \varphi) \ni p$, $(V, \psi) \ni q = f(p)$ in N such that $\psi \circ f \circ \varphi^{-1}$ is a canonical immersion.

2. If $f : M \to N$ is a submersion at $p \in M$. Then $\exists$ charts $(U, \varphi) \ni p$, $(V, \psi) \ni q = f(p)$ such that $\psi \circ f \circ \varphi^{-1}$ is a canonical submersion.

<u>Proof</u>: 1. Choose chart $(U, \varphi) \ni p$ such that $x_i(p) = 0$ for $1 \leq i \leq m = \dim M$ and $(V, \psi) \ni q = f(p)$ with coords y_i, $y_i(q) = 0$, $1 \leq i \leq n = \dim N$.

- Choose a basis $\{v_1, \dots, v_m\}$ of $T_p M$ and $\{w_i\}_{1 \leq i \leq n}$ of $T_q N$ such that $Df(p) v_i = w_i$ $(v_i \mapsto w_i)$ for $1 \leq i \leq m$. Let $G := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$.
- May assume by modifying basis of $\mathbb{R}^m$, $\mathbb{R}^n$ if nec. that $DG(\varphi(p))$ is canonical : $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

$\rightarrow$

• Let $H : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ be the map $H(x, z) = G(x) + (0, z)$, where $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \ni (0, z)$.

Then (?) $DH_{(p,0)} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity.

$\Rightarrow$ H is a local diffeomorphism at $(p, 0)$. Hence the neighbourhoods $(U, \varphi) \ni p$ and $(V, H \circ \psi^{-1})$ do the job. (somethings to check).

2. Here we assume $Df(p)$ is onto: $T_pM \to T_{f(p)}N$, So has full rank, $m \geq n$. Hence we may assume that the Jacobian matrix of f at p has its first $n \times n$ minor invertible, so

$\left(\frac{\partial f_j}{\partial x_i}(p) \right)_{1 \leq i,j \leq n}$ is invertible. Consider $\eta : V \to \mathbb{R}^n$

by $\eta(z) = (f_1(z), \dots, f_n(z))$, $z \in V$; f_i being the coordinate functions of f on V.

• Then η is a <u>local</u> diffeom at $q = f(p)$.

The charts $(U, \varphi) \ni p$ & $(W, \eta|_W) \ni q$ do the job, where $\eta|_W$ is a diffeo, $W \subseteq V$.

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Next: discuss submanifolds using the above discussed notions, examples etc.