

Lecture-1

Prerequisites: DG-1 (Curves & Surfaces in $\mathbb{R}^3$),
Multivariable calculus, Linear algebra
(tensor products, eigenvalues etc).

Differential geometry aims at studying "differentiable" properties of geometric objects e.g. Curves, Surfaces or more generally manifolds, as one of its goals, by studying smoothness of functions between two such objects; by developing a Calculus; one needs to worry about "integrating" functions in a suitable sense, which we need for variety of reasons.

To make this somewhat more precise, given a smooth (i.e. without 'spikes') object, we study its shape by studying how it curves around various points.

This job is simpler if we study such objects as subobjects of some Euclidean Space, i.e. embedded suitably in some $\mathbb{R}^n$, by taking of tangent spaces at points and how they

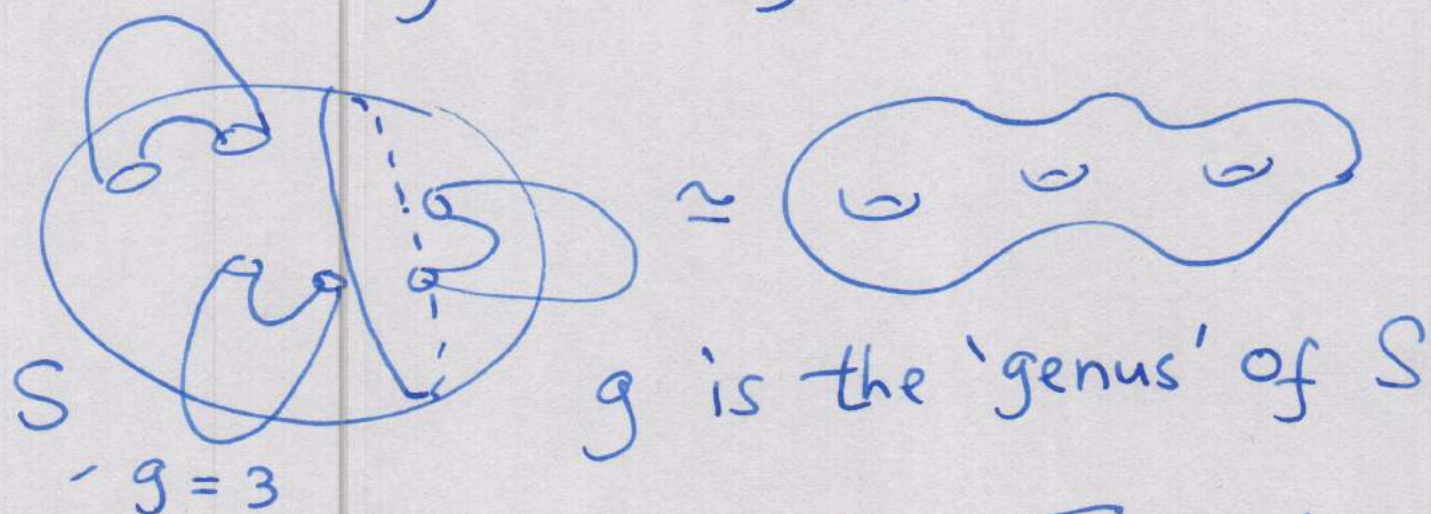
² Change while passing from one point to the other; leading to the notion of "normal curvature". This is one way of measuring how the given object curves around various points.

Another way is to use the notion of curvature for 'plane' curves and consider curvatures of various curves on the surface, through a point, obtained by dissecting the surface by a normal plane at that point, containing a fixed tangent direction; this leads to "geodesic" curvature. One encounters more notions of curvatures for a surface and all contribute to determining the surface (locally). However, all these notions assume that the surface is contained in $\mathbb{R}^3$, i.e. we always have an embedding in picture. If $S \subseteq \mathbb{R}^3$ be a surface and assuming we have a reasonable function $K: S \rightarrow \mathbb{R}$ giving curvature $K(p)$ at $p \in S$, such that if $S = \text{plane}$, then $K(p) = 0 \forall p$; $S = S^2$ then K is a constant function etc.

³ and assuming we have defined a suitable notion of $\int$ on S , then we obtain a fantastic result

$$\int_S K dA = 2\pi \chi(S)$$

Where dA is the area element, $\chi(S)$ is the "Euler characteristic" of S , where S is assumed compact. From topology one knows S orientable, is obtained from S^2 by attaching g -handles.



and $\chi(S) = 2 - 2g$. The above result can be interpreted as giving the 'total curvature' of S in terms of a topological invariant of S !!

Just to remind you, topology does not regard

⁴ a 'deformed' version of S as different, as long as the deformation does not tear S (only 'stretches' or 'bends'). This is why the above result is amazing; it says that even after any deformation, the total curvature of S remains the same.

Going beyond: As you may have guessed, this course is about objects beyond curves & surfaces, i.e. (smooth) manifolds. One can ask at this point: Why study manifolds? Of course, an easy & defensive answer is that manifolds of higher dimensions occur in physics, e.g. in relativity theory & so on. There may be sources of geometric examples, however manifolds do occur in branches of math as well, perhaps not so 'geometrically'. Let us see some such examples:

- Let $M_{m,n}(\mathbb{R})$ be the space of all $m \times n$ matrices with entries from $\mathbb{R}$. We identify this with $\mathbb{R}^{mn}$ to topologize it. $\leadsto$

^s Then, for k , the set

$$M_{m,n;k}(\mathbb{R}) = \{A \in M_{m,n}(\mathbb{R}) \mid \text{rank } k\}$$

is a C^∞ -manifold (we'll discuss this later).

- Grassmann manifolds: you may have come across projective spaces $\mathbb{P}_{\mathbb{R}}^n$; this is the set of all 1-dim'l subspaces of $\mathbb{R}^{n+1}$, with a natural topology.

The Grassmann manifold $G_k(\mathbb{R}^n)$ consists of all k -dim'l subspaces of $\mathbb{R}^n$. We can give this a natural manifold structure.

So $\mathbb{P}_{\mathbb{R}}^n = G_1(\mathbb{R}^{n+1})$. We'll see that

$$\cancel{M_{m,n;k}^{M_{k,n}}(\mathbb{R})} \quad \simeq \quad G_k(\mathbb{R}^n), \text{ where}$$

$\sim$

for $x, y \in M_{k,n}(\mathbb{R}) = \cancel{M_{k,n}}^{\cancel{M_{k,n}}}$, $x \sim y$ if

$\exists C \in GL_k(\mathbb{R})$ such that $X = CY$.

⁶ These two examples are extremely useful mathematically, yet they do not appear in nice, geometrically speaking, shape. However, by investigating geometric notions about such 'abstract' spaces, one arrives at concrete results that are useful.

→ Another class of examples comes from Lie groups, which occur naturally in Physics and Math, and what are called 'homogeneous spaces'.

When our surface is embedded in $\mathbb{R}^n$, we can talk of the normal at a point, since we have $\langle, \rangle$ on $\mathbb{R}^n$. How do we do this for manifolds as above?

This is necessary in defining ideas of curvature etc. The idea of tangent space at a point is the first hurdle to cross; for a manifold M and $p \in M$, to define $T_p(M)$, the tangent space to

7. to M at p . Once that is done, we can define an inner product on $T_p(M)$, say $\langle \cdot, \cdot \rangle_p$, for all $p \in M$ is a 'smooth' way. Using this, we can overcome some of the difficulties.

Using tangent spaces, we can talk of vector fields on M , or more generally, tensor fields on M . These will be defined intrinsically, without embedding M in any Euclidean space. Once we have overcome the technical hurdles and defined notion of curvature, one may ask if we have a Gauss-Bonnet like theorem; investigate relations with vector fields (on S^2 , which has constant $\neq 0$ curvature, there is no nowhere vanishing vector field).

→ The topology of a manifold itself poses challenges. To give an example, →

8. Using curves on M , we may define a metric on M , natural enough to M . How is that related to the topology of M ? Using curves on M , we may talk of geodesics on M ("straight" curves on M) and study those M where any two points can be joined by a geodesic. Can we relate this property to M being complete, e.g. in the above metric? This is a fundamental result known as the Hopf-Rinow theorem, giving an affirmative answer in the case of M compact. One proves that when M is smooth, simply connected and has non positive 'sectional' curvature, M is diffeomorphic to $\mathbb{R}^n$. This course will aim to prove such landmark results as well as the classical Stokes theorem. Initial parts will develop the technical machinery that will be needed $\rightarrow$ Starting from next Class!

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Lecture-2

A review of calculus: Though the basic objects of study in this course are manifolds, these are locally Euclidean, i.e. every point has a neighbourhood $\simeq$ an open ball in $\mathbb{R}^n$. This allows us to do calculus on such abstract objects, using what are called chart maps and transition maps. We therefore recall first the calculus on $\mathbb{R}^n$: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} =: \lambda$ exists; then we define the derivative f' of f at a to be λ , i.e. $f'(a) = \lambda$.

→ The derivative of f at a represents the best linear approximation of f at a .

The above may be rephrased as: $f: \mathbb{R} \rightarrow \mathbb{R}$ is diff at a with derivative $\lambda = f'(a)$ if $\lim_{x \rightarrow a} \frac{f(x) - f(a) - \lambda \cdot (x - a)}{x - a} = 0$.

→

² Since any $\mathbb{R}$ -linear map $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is of the form $\phi(x) = \alpha x$ for some $\alpha \in \mathbb{R}$ fixed, we may think of $\lambda \cdot (x-a)$ above as $\phi(x-a)$ for $\phi: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \lambda \cdot x$. This leads & motivates us to define

Defⁿ: A map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if $\exists$ a linear map $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that
$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0$$

(Note that the norms on numerator & denominator are in $\mathbb{R}^m$ & $\mathbb{R}^n$ resp.).

Exercise 1: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be diff at $a \in \mathbb{R}^n$ and $T_i: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear maps, $i=1,2$, such that
$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - T_i(h)\|}{\|h\|} = 0$$

for $i=1,2$; then $T_1 = T_2$.

2. For sufficiently small $x \in \mathbb{R}^n$ we have
$$f(a+x) = f(a) + \lambda(x) + \|x\| \epsilon(x),$$
 where $\epsilon(x)$ satisfies $\lim_{x \rightarrow 0} \epsilon(x) = 0$.

Notation :- The linear map λ as above, if it exists, is denoted by $Df(a)$ and is called the derivative of f at a .

So $Df(a)$ is a linear transformation
 $: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Exercise : For any $x \in \mathbb{R}^n$,

$$Df(a)(x) = \lim_{t \rightarrow 0} \frac{1}{t} [f(a+tx) - f(a)].$$

Computing $Df(a)$:- For this, we'll compute

the matrix of $Df(a) \in M_{m \times n}(\mathbb{R})$, where

we write $Df(a) = [C_1 \cdots C_n]$, C_i are the columns of $Df(a)$, and $C_i := Df(a)(e_i)$.

By the above exercise

$$Df(a)(e_j) = \lim_{t \rightarrow 0} \frac{1}{t} [f(a+te_j) - f(a)]$$

→ We can write $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ in terms of its coordinate functions $f = (f_1, \dots, f_m)^t$,

$f_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $1 \leq i \leq m$; so $f = \sum_{j=1}^m f_j e_j'$,

$$e_j' = (0, \dots, 0, \overset{j^{\text{th}}}{1}, 0, \dots, 0)^t.$$

Hence we have $\rightsquigarrow$

$$^4 \quad f(a + te_j) - f(a) = \begin{pmatrix} f_1(a + te_j) - f_1(a) \\ \vdots \\ f_i(a + te_j) - f_i(a) \\ \vdots \\ f_m(a + te_j) - f_m(a) \end{pmatrix}$$

and $f_i(a + te_j) - f_i(a)$, for $a = \sum_{k=1}^n a_k e_k$

$$= f_i((a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n)^t)$$

$$- f_i((a_1, \dots, a_j, \dots, a_n)^t).$$

i th row of the j th column of $Df(a)$ is

$$\lim_{t \rightarrow 0} \frac{f_i((a + te_j)) - f_i(a)}{t} = (ij)^{\text{th}} \text{ entry of}$$

the matrix of $Df(a)$.

Defⁿ (Partial derivatives): For $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$

and $a \in \mathbb{R}^n$, the partial derivative of f at $a = (a_1, \dots, a_n)^t$ with respect to x_i (the i th coordinate) is the limit

$$\frac{\partial \varphi}{\partial x_i} \Big|_a = \lim_{t \rightarrow 0} \frac{\varphi((a_1, \dots, a_{i-1}, a_i + t, a_{i+1}, \dots, a_n)) - \varphi((a_1, \dots, a_n))}{t},$$

if it exists. We therefore have



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$$Df(a) = (a_{ij}) \text{ where } a_{ij} = \frac{\partial f_i}{\partial x_j} \Big|_a.$$

This matrix is called the Jacobian matrix of f at a .

Examples :- Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a diff. funcⁿ.

The derivative of f at $p \in \mathbb{R}^n$, $Df(p)$ is a $1 \times n$ matrix, i.e. a row vector, called the gradient of f at p .

$$(\nabla f)|_p = \left(\frac{\partial f}{\partial x_1} \Big|_p, \dots, \frac{\partial f}{\partial x_n} \Big|_p \right).$$

The level sets of f are the sets $f^{-1}(c)$, $c \in \mathbb{R}$. For generic c , these give hyper-surfaces in $\mathbb{R}^n$, i.e. geom. objects having $\dim n-1$, $\subseteq \mathbb{R}^n$. Since f is differentiable, for generic c , these objects are 'differentiable'.

To make sense of this, assume that $p \in S := f^{-1}(c)$ is such that $\rightarrow$

⁶ $\nabla f|_p \neq 0$. Then $\nabla f|_p$ would constitute a normal at p to the hyper-surface $S \subseteq \mathbb{R}^n$. If $\nabla f|_p \neq 0$, we can define

$$T_p S = \left\{ (b_1, \dots, b_n) \in \mathbb{R}^n \mid \langle \nabla f|_p, (b_1 - a_1, \dots, b_n - a_n) \rangle = 0 \right\}$$

where $p = (a_1, \dots, a_n)$.

$$= \left\{ (b_1, \dots, b_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{\partial f}{\partial x_i} \Big|_p \cdot (b_i - a_i) = 0 \right\},$$

the tangent space to S at p .

Remark:- There are examples (?) of $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\frac{\partial f_i}{\partial x_j} \Big|_a$ exist $\forall i, j$, yet f is not continuous at a .

$\rightarrow f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is diff at $a \Rightarrow f$ is cont. at a .

Thm:- $Df(a)$ exists if all $\partial f_i / \partial x_j$ exist at all points in an open nbd of a and if each $\partial f_i / \partial x_j$ is continuous at a . $\square$

7 Chain rule :- If $f: U \rightarrow \mathbb{R}^m$ and $g: V \rightarrow \mathbb{R}^l$,
 Where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open and
 $f(U) \subseteq V$; f is diff at $a \in U$, g is diff
 at $f(a) \in V$; then the composite
 $g \circ f: U \rightarrow \mathbb{R}^l$ is diff. at a and its
 derivative at a is given by

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

$$\mathbb{R}^n \xrightarrow{Df(a)} \mathbb{R}^m \xrightarrow{Dg(f(a))} \mathbb{R}^l$$

$$Dg(f(a)) \circ Df(a) = D(g \circ f)(a)$$

→ If $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an affine map,
 i.e. $\exists T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear & $v \in \mathbb{R}^m$
 with $L(x) = T(x) + v \quad \forall x \in \mathbb{R}^n$; Then
 L is differentiable and $DL(a) = T \quad \forall$
 $a \in \mathbb{R}^n$.

→ $f, g: U \rightarrow \mathbb{R}^m$ be diff at $a \in U \subseteq \mathbb{R}^n$;
 U open;

$\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is diff at a .

→ $f, g: U \rightarrow \mathbb{R}$ diff at $a \in U \Rightarrow f \cdot g$ diff at a .

If $f(x) \neq 0 \quad \forall x \in U$, then $x \mapsto f(x)^{-1}$ is
 also diff at a . →

8.

→ Hence any polynomial function $P: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at all points; a rational function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $f(x) = \frac{P(x)}{Q(x)}$ where P & Q are polynomials, is differentiable on the open set $U = \mathbb{R}^n - Z(Q)$, $Z(Q) = \{x \in \mathbb{R}^n \mid Q(x) = 0\}$.

Exercise :- 1. Identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Then

$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is open. Prove that $i: GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ $x \mapsto x^{-1}$ is differentiable at all points in $GL_n(\mathbb{R})$.

2. Consider $f: \mathbb{R}^2 \rightarrow \mathbb{R}; (x, y) \mapsto \begin{cases} \frac{x|y|}{(x^2+y^2)^{3/2}}, & (x, y) \neq (0, 0) \\ 0, & x=y=0. \end{cases}$

show that f is not diff at $(0, 0)$, both partial derivatives exist at $(0, 0)$.

• show that restriction of f to every line passing through $(0, 0)$ is diff.

_____ -x _____ -x _____ -x _____ →

Lecture-3

Calculus contd :- The class C^r, C^∞ etc. We saw that the existence and continuity of the partial derivatives of a function f on an open $U \subseteq \mathbb{R}^n$ implies differentiability of f at all points of U ; If $f: U \rightarrow \mathbb{R}^m$ is differentiable on U , we have $Df: U \rightarrow \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ $a \mapsto Df(a)$. We identify $\text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ with the Euclidean space $\mathbb{R}^{mn}$ as usual. We say f is of class C^1 or f is C^1 if the map $a \mapsto Df(a)$ as above is continuous. Inductively, we define f to be C^r if Df is of class C^{r-1} ($C^0 \equiv$ continuous maps).

→ If f is $C^r \forall r \geq 1$, we say f is C^∞ or smooth.

Exercise :- $f \in C^r \Rightarrow$ all mixed partials

$$D^\alpha f_i := \frac{\partial^{i_1} \partial^{i_2} \dots \partial^{i_n}}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_n^{i_n}} f_i \text{ exist and are}$$

Continuous $\forall 1 \leq i \leq m \wedge \forall \alpha = (i_1, \dots, i_n)$

with $|\alpha| := \sum_j i_j \leq r$; Conversely this $\Rightarrow f \in C^r$.

Diffeomorphisms :- Fix $1 \leq r \leq \infty$. A C^r map

$f: U \rightarrow V$, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, open, is called a C^r -diffeomorphism if $\exists$ a C^r map $g: V \rightarrow U$ such that $g \circ f = 1_U$, $f \circ g = 1_V$.

A C^∞ -diffeomorphism is called smooth. When $r=0$, the notion simply means f is a homeomorphism.

→ If $f: U \rightarrow V$ is a C^r diffeomorphism, then the Jacobian matrix of f is invertible ($r \geq 1$) pointwise (why?). Note that if $\exists$ a C^r diffeomorphism $U \rightarrow V$, then $n=m$. This also holds when $r=0$. This follows from "Brouwer's invariance of domain" theorem from algebraic topology.

Exercise :- Let $r \geq 1$ and $f \in C^r; U \rightarrow V$ and assume f is a C^1 -diffeomorphism. Then f is a C^r -diffeomorphism.

Local diffeomorphisms :- Let $U \subseteq \mathbb{R}^n$ open & $f: U \rightarrow \mathbb{R}^n$ be a C^r map. We call f a C^r -local diffeomorphism at $p \in U$ if $\exists$ an open nbhd W of p such that $f|_W: W \rightarrow f(W)$ is a diffeomorphism.

$z \mapsto e^z$ is a C^r -diffeom.

Remark: If f is a C^r -local diffeom for all $p \in U$, then also f has invertible Jacobian at all $p \in U$; here $r \geq 1$.

Example:- Consider $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$(x, y) \mapsto (e^x \cos y, e^x \sin y).$$

(Equivalently $z \mapsto e^z: \mathbb{C} \rightarrow \mathbb{C}$). This is not a diffeom (Why?), but f restricted to any open strip $\mathbb{R} \times (b, b+2\pi)$ is a diffeom. Note that f maps this open strip (diffeomorphically) to the half-slit plane, i.e. $\mathbb{R}^2$ minus the half ray $\{(\lambda \cos b, \lambda \sin b); \lambda \geq 0\}$ (picture this?).

Exercise:- $U, V \subseteq \mathbb{R}^n$ be open, $f: U \rightarrow V$ a smooth bijection. Then f is a smooth diffeo $\Leftrightarrow f$ is a C^1 -local diffeo at all $p \in U$.

Inverse function theorem:- Let $U \subseteq \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}^n$ be C^r , $r \geq 1$. Let $p \in U$ and $Df(p)$ be invertible. Then f is $\rightarrow$

3-4 a C^r -local diffeom at p .

(Sketch of) proof :- First suppose f is an

affine map, i.e. $f(x) = Ax + v$ where

$A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear and $v \in \mathbb{R}^n$ is a fixed vector. Then, for $w, z \in U$, we have

$$f(w) - f(z) = Aw - Az = Df(z)(w - z) \quad (?).$$

Hence if $Df(z) = A$ is invertible and we wish to have the inverse image w of some $y = f(w)$, we can solve it as

$$\boxed{w = z - Df(z)^{-1}(f(z) - y)} \quad \sim \#$$

→ Let f be any map satisfying the hypothesis

Using $\varphi(x) := Df(p)^{-1}(f(x+p) - f(p)): U \rightarrow \mathbb{R}^n$

we may assume (without loss of generality)

that $p=0$, $f(0)=0$ and $Df(0) = I_{\mathbb{R}^n}$

→ Let $B = \overline{B(0, \delta)}$ be the closed ball of radius δ around the origin s.t. $\overline{B} \subseteq U$ (δ to be chosen suitably). For $y \in \mathbb{R}^n$

define $T_y: B \rightarrow \mathbb{R}^n$ by $z \mapsto z - (f(z) - y)$,

motivated by $\#$, so that this would give the inverse image of y when f is affine.

3-5
Lemma-1 :- Let $U \subseteq \mathbb{R}^n$ be an open ball around the origin, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C^1 and suppose $\exists M > 0$ such that $|\partial f_i / \partial x_j (x)| \leq M$ $\forall x \in U$. Then $\|f(x) - f(y)\| \leq n^2 M \|x - y\|$ $\forall x, y \in U$.

Lemma-2 (Contraction mapping principle).
 Let (X, d) be a complete metric space, $f: X \rightarrow X$ be a contraction, i.e., $d(f(x), f(y)) \leq \lambda d(x, y)$ for some $0 \leq \lambda < 1, \forall x, y \in X$.
 Then $\exists!$ $x \in X$ such that $f(x) = x$.

The function $g: \mathbb{R}^n \rightarrow \mathbb{R}^n, g(x) = x - f(x)$.
 Then $g(0) = 0$ & $Dg(0) = 0$; g is C^1
 and $\exists \delta > 0$ s.t.

$$\left| \frac{\partial g_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(0) \right| < \frac{1}{2n^2} \quad \forall x \in B(0, \delta)$$

$$\left| \frac{\partial g_i}{\partial x_j}(x) \right| \leq \frac{1}{2n^2} \quad \text{Use Lemma-1}$$

$$\& \text{ put } y=0 \text{ to get } \|g(x)\| < \frac{\|x\|}{2} \quad \forall x \in B(0, \delta)$$

$$\therefore g(B(0, \delta)) \subseteq B(0, \delta/2).$$

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$$\rightarrow T_y(x) = x - (f(x) - y) = y + g(x).$$

$$\Rightarrow \|T_y(x)\| \leq \|y\| + \|g(x)\|. \text{ Hence}$$

$$\text{for } y \in \overline{B(0, \delta/2)}, \|T_y(x)\| \leq \delta/2 + \delta/2 = \delta$$

$$\text{and so } T_y: \overline{B(0, \delta)} \rightarrow \overline{B(0, \delta)} \text{ and}$$

$$T_y(x) = x \Leftrightarrow y + x - f(x) = x \Leftrightarrow y = f(x).$$

$$\rightarrow f \text{ is 1-1 on } \overline{B(0, \delta)} \Leftrightarrow T_y \text{ has a!}$$

$$\text{fix-}\omega \text{ point in } \overline{B(0, \delta)}.$$

$$\rightarrow \text{For } y \in \overline{B(0, \delta/2)}, T_y \text{ is a contraction.}$$

$$\rightarrow \text{Hence } f \text{ is 1-1 on } \overline{B(0, \delta)}.$$

$$\rightarrow f(\underbrace{B(0, \delta/4)}_W) \text{ is open; } \tilde{f}: f(W) \rightarrow W$$

$$\text{is smooth: } \forall p \in W, f \in C^1 \Rightarrow$$

$$f(x) - f(p) = Df(p)(x-p) + e(x, p),$$

$$\text{Where } \lim_{x \rightarrow p} \frac{e(x, p)}{\|x-p\|} = 0. \quad (*)$$

$$\text{Hence if } y = f(x) \text{ \& } q = f(p), A = Df(p)^{-1},$$

$$\text{we have } \rightsquigarrow$$

$$3-7 \quad A(y-q) = \bar{f}'(y) - \bar{f}'(q) + A \in (\bar{f}'(y), \bar{f}'(q)).$$

$$\text{Hence } \bar{f}'(y) - \bar{f}'(q) = A(y-q) + \epsilon'(y, q).$$

$$\text{Now } \lim_{y \rightarrow q} \frac{\epsilon'(y, q)}{\|y - q\|} = -A \lim_{y \rightarrow q} \underbrace{\epsilon(\bar{f}'(y), \bar{f}'(q))}_{\rightarrow 0(?)} \text{ and}$$

Hence $\bar{f}'$ is differentiable $\forall p \in W$ &

$$(\mathbb{D}\bar{f}')(p) = (\mathbb{D}f)(\bar{f}'(p))^{-1} \quad \square$$

- The IFT controls the local behaviour of a function, i.e. says what happens in a nbhd of a point, from the information on its derivative at that point.
- It is possible (?) for a map to be a local diffeomorphism at each point without being a global diffeomorphism. However any $f: \mathbb{R} \rightarrow \mathbb{R}$ that is a C^r -local diffeomorphism at each point (here $r \geq 1$) is indeed a global diffeomorphism.

(?).

Exercise 1. In general $C^{r+1} \subsetneq C^r$.

2. $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}$ has non-vanishing derivative $Df(0) = 1/2$ at $x=0$. $\rightarrow$

38 However, f is not a diffeom in any nbhd of $x=0$; in fact f even fails to be 1-1 in any nbhd of 0. This shows that f being at least C^1 in IFT cannot be relaxed.

One uses IFT to prove the
Implicit function Theorem: Let $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be continuously differentiable in an open set containing $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ and $f(a, b) = 0$. Let M be the $m \times m$ matrix $\left(\frac{\partial f_i}{\partial x_{n+j}}(a, b) \right)_{1 \leq i, j \leq m}$. If $\det M \neq 0$, there exists an open set $V \subseteq \mathbb{R}^n$, $a \in V$; an open $W \subseteq \mathbb{R}^m$, $b \in W$, such that for every $x \in V$, $\exists!$ $g(x) \in W$ (with $g(a) = b$), $f(x, g(x)) = 0 \forall x \in V$. The function $g: V \rightarrow W$ thus defined is differentiable. $\square$

In the next class, we'll see some applications of these results. It cannot be emphasized enough that these results are very fundamental to differential geometry.

—————x—————x—————x—————x—————x—————x—————x—————

In the last lecture we discussed path connectedness of some Lie groups, the case of $SL(n; \mathbb{R})$ was particularly interesting. A similar proof, with $A \in GL(n; \mathbb{C})$, $A^* = A$ define as positive if $\langle x, Ax \rangle > 0 \quad \forall x \in \mathbb{C}^n - \{0\}$, where $\langle, \rangle$ is the (sesquilinear) inner product on $\mathbb{C}^n$ gives

Theorem :- (Polar decomposition for $SL(n; \mathbb{C})$): Given $A \in SL(n; \mathbb{C}) \exists$ a unique pair (U, P) with $U \in SU(n)$, $P = P^*$ and P positive, with $A = UP$; $\det P = 1$.

Exercise :- Polar decompositions for $GL(n; \mathbb{R})$, $GL(n; \mathbb{R})^+$ and $GL(n; \mathbb{C})$.

Let us now move towards the notion of Lie algebra for a Lie group G . Remember that G is a smooth manifold, with all points behaving alike, via the left/right translation maps, which are smooth.

Roughly speaking the Lie algebra of G consists of all smooth vector fields on G that are (left) translation invariant. Such fields are completely determined by their value at the identity.

³⁻² element of G . Moreover, if X is such a vector field and $e \in G$ is the identity element, $a \in G$ any other point, to get $X(a)$, we use the translation by a and invariance of X . Given any element v of $T_e(G)$, the tangent space of G at e , one can construct a smooth vector field on G by translating v around on G . We need thus focus on $T_e(G)$ as far as left invariant vector fields on G are concerned. Recall the notion of an integral curve for a vector field. For a smooth manifold M and a smooth vector field X on M , an integral curve of X is a smooth curve $\alpha: (a, b) \rightarrow M$ such that the tangent vector to α at $t \in (a, b)$ is exactly the element $X(\alpha(t)) \in T M$, $\forall t \in (a, b)$. These curves are extremely useful for $\alpha(t)$ calculus on M . For a matrix Lie group, the exponential map gives a uniform recipe for such curves.

Exponential map: - We identify $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$ and, for $x \in M_n(\mathbb{C})$, $\|x\| := \sqrt{\sum_{k=1}^n |x_{kk}|^2}$

Then $\|x+y\| \leq \|x\| + \|y\|$ and $\|xy\| \leq \|x\| \|y\|$ $\forall x, y \in M_n(\mathbb{C})$. We can talk of convergence of sequences of matrices as usual; $X_m \rightarrow X$ iff $\|X_m - X\| \rightarrow 0$. Cauchy sequences are defined. $\leadsto$

It follows easily that $M_n(\mathbb{C})$ with this metric is complete, i.e. every Cauchy sequence converges.

An infinite series $\sum_{i=0}^{\infty} x_i$ is defined to converge absolutely if the series $\sum_{i=0}^{\infty} \|x_i\|$ converges.

If this is the case, then the sequence $(S_m)_m$ of partial sums $S_m = x_0 + \dots + x_m$ is a Cauchy sequence, hence has a well defined limit, which we define as the sum $x_0 + x_1 + \dots$.

Proposition 1: For any $x \in M_n(\mathbb{C})$, the series

$$e^x := \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad \text{converges. The map}$$

$x \mapsto e^x$ on $M_n(\mathbb{C})$ is continuous.

Proof: We have $\sum_{i=0}^{\infty} \left\| \frac{x^i}{i!} \right\| \leq \sum_{i=0}^{\infty} \frac{\|x\|^i}{i!} = e^{\|x\|}$

So the series e^x is absolutely convergent, hence is convergent.

Clearly each partial sum is a continuous funcⁿ of x , the series of e^x converges uniformly on closed balls $\{\|y\| \leq r\}$. Hence $x \mapsto e^x$ is continuous. $\square$

Some properties of e^X : Let $X, Y \in M_n(\mathbb{C})$.

(i) $e^0 = I$ (ii) $(e^X)^* = e^{X^*}$ (iii) $e^X \in GL_n(\mathbb{C})$,

$(e^X)^{-1} = e^{-X}$ (iv) $e^{(a+b)X} = e^{aX} e^{bX} \quad \forall a, b \in \mathbb{C}$.

(v) If X, Y commute then $e^{X+Y} = e^X e^Y = e^Y e^X$.

(vi) For any $A \in GL_n(\mathbb{C})$, $A e^X A^{-1} = e^{A X A^{-1}}$.

(vii) $\|e^X\| \leq e^{\|X\|}$.

Let us prove some of these: (i) is clear.

For (ii) Let $S_{m,*} := 1 + X^* + \frac{X^{*2}}{2!} + \dots + \frac{X^{*m}}{m!}$.

Then $S_{m,*} = S_m^*$, where $S_m = 1 + X + \dots + \frac{X^m}{m!}$.

Taking limits the result follows, as $X \mapsto X^*$ is continuous.

(v) Let $XY = YX$ and consider

$$e^X e^Y = \left(I + X + \frac{X^2}{2!} + \dots \right) \left(I + Y + \frac{Y^2}{2!} + \dots \right)$$

To compute the RHS, we may multiply the series term by term (?). Collecting total degree m terms we get

$$e^X e^Y = \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{X^k}{k!} \frac{Y^{m-k}}{(m-k)!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^m \frac{m!}{k!(m-k)!} X^k Y^{m-k}$$



3-5 Since $XY = YX$, we have the expansion

$$(X+Y)^m = \sum_{k=0}^m \frac{m!}{k!(m-k)!} X^k Y^{m-k}, \text{ so}$$

$$e^X e^Y = \sum_{m=0}^{\infty} \frac{1}{m!} (X+Y)^m = e^{X+Y}.$$

(vi) Follows since $(AXA^{-1})^i = AX^iA^{-1}$.

(vii) Left to you! □

Coming back to the integral curves for vector fields, using the matrix exponential, we have

Theorem-3: Let $X \in M_n(\mathbb{C})$. Then $t \mapsto e^{tX}$ defines a smooth curve in $M_n(\mathbb{C})$. Moreover,

$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X. \text{ Hence,}$$

$$\left. \frac{d}{dt} e^{tX} \right|_{t=0} = X.$$

Proof: We can differentiate the power series e^{tX} term by term, since for $1 \leq i, j \leq n$, $(e^{tX})_{ij}$ is a convergent power series in t etc. □

Our next job is to get expression for e^X . We first do this for X diagonalizable, then

³⁻⁶ nilpotent and then for general X using what is known as the Jordan decomposition for X

$$X = S + N, \quad S \text{ diagonalizable, } N \text{ nilpotent,}$$

$SN = NS$; this decomposition is unique such. (we will discuss this later).

1. Diagonalizable: Let $A \in GL(n; \mathbb{C})$ such that $A X A^{-1} = D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$e^{A X A^{-1}} = A e^X A^{-1} = e^D = \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}). \text{ So}$$

$$e^X = A^{-1} \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) A.$$

2. Nilpotent: In this case $X^m = 0$ for some $m > 0$. But then e^X is a polynomial in X .

3. General case: Write $X = S + N$, S diagonalizable, N nilpotent and $NS = SN$. Then it follows that

$$e^X = e^{S+N} = e^S e^N$$

We have the recipe for e^S as well as e^N , so we have found an expression for e^X .

In the next class we will see some more properties of the matrix exponential and proceed towards the notion of Lie algebra of a Lie group.



Lecture - 4

Remarks on the Implicit Function Theorem :- To understand what the above achieves, consider a relation, say in $\mathbb{R}^2$, given as $F(x, y) = 0$. Such a "rel" does not necessarily represent a function (of say x), i.e. this may not lead to a solution of the form $y = f(x)$ for a function f (e.g. $F(x, y) = x^2 + y^2 - 3$). One naturally asks: When, a "rel" $F(x, y) = 0$, represents a function, or equivalently, when can the eqⁿ $F(x, y) = 0$ be solved explicitly for y in terms of x , giving a! solution?

The Implicit function theorem tells us that 'locally' this can be achieved (for good F), i.e. given a point (x_0, y_0) with $F(x_0, y_0) = 0$, under 'certain conditions', $\exists$ a nbd of (x_0, y_0) such that in this nbd, $F(x, y) = 0$ yields a functional solⁿ $y = f(x)$. The conditions stipulate that F & $\partial F / \partial y$ be continuous in some nbd of (x_0, y_0) and $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.

→ More generally: let $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be cont. diff. in an open nbd of $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$, $f((a, b)) = 0$. Assume that the $m \times m$ matrix $\frac{\partial f}{\partial y}(a, b) \neq 0$.

4-2 $\left(\frac{\partial f_i}{\partial x_{n+j}}(a,b) \right)_{1 \leq i,j \leq m}$ is invertible. Then

$\exists$ a nbd $A \times B \subseteq \mathbb{R}^n \times \mathbb{R}^m$ of (a,b) and a diff. function $g: A \rightarrow B$, $g(a)=b$, s.t. $f(x, g(x)) = 0 \quad \forall x \in A$. $\square$

Submersion: A C^r -map, for $r \geq 1$, $f: U \rightarrow \mathbb{R}^m$ is a submersion at $a \in U$ (open) if $Df(a)$ is surjective. (This $\Rightarrow n \geq m$ and Jacobian of f at a has rank m). $\square$

Immersion: Let $r \geq 1$. A C^r -map $f: U \rightarrow \mathbb{R}^m$ is a C^r -immersion at $a \in U$ if $Df(a)$ is injective; (so in this case $n \leq m$ and the Jacobian of f at a has rank n).

Rank theorem: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be cont. diff. in an open set $\ni a$, $n \geq m$. If $f(a)=0$ and the $m \times n$ matrix $\left(\frac{\partial f_i}{\partial x_j}(a) \right)$ has rank m ; then $\exists$ an open set $A \subseteq \mathbb{R}^n$ and a diff. funcⁿ $h: A \rightarrow \mathbb{R}^n$ with a diff inverse (i.e. h is a diffeom) s.t. $f \circ h(x_1, \dots, x_n) = (x_{n-m+1}, \dots, x_n)$. $\square$

Hence one can modify the coordinates around the point a by the function h to get a simpler form for f locally.

Level sets: Let $U \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ be open & $f: U \rightarrow \mathbb{R}$ be smooth. Let $c \in \mathbb{R}$ be such that for $S = f^{-1}(c)$, $(\nabla f)(p) \neq 0$ $\forall p \in S$ (we assume such c exists for f). We then call S a hypersurface in $\mathbb{R}^{n+1}$.

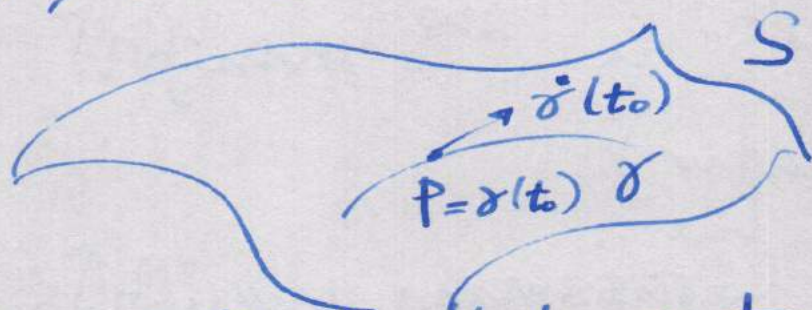
A parametrized curve in $\mathbb{R}^{n+1}$ is a smooth function $\gamma: I \rightarrow \mathbb{R}^{n+1}$, I some open interval in $\mathbb{R}$; so $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+1}(t))$, $\gamma_i: I \rightarrow \mathbb{R}$ are smooth. Denote

$$\dot{\gamma}(t) := (\dot{\gamma}_1(t), \dots, \dot{\gamma}_{n+1}(t)), \text{ where } \dot{\gamma}_i(t) = \frac{d}{dt} \gamma_i(t). \text{ When } \dot{\gamma}(t_0) \neq 0, \text{ we}$$

call $\dot{\gamma}(t_0)$ the tangent vector to $\gamma(t)$ at the point $\gamma(t_0)$. (justify this?).

We define, for S as above $\leadsto$

4-4 a tangent vector to S at $p \in S$ is,
 by defⁿ, $\dot{\gamma}(t_0)$ for some curve $\gamma \in S$,
 where $p = \gamma(t_0)$, i.e. γ lies on S and
 passes through p .



We denote $T_p(S)$ = set of all tangent
 vectors to S at p . We first observe

Lemma :- The gradient of f at p , i.e.
 $\nabla f(p)$ ($\equiv Df(p)$) is orthogonal to
 $T_p(S)$.

Proof :- Let $v \in T_p(S)$. Then $v = \dot{\gamma}(t_0)$
 for some $\gamma: I \rightarrow S = f^{-1}(c)$. Therefore
 $f(\gamma(t)) = c \quad \forall t \in I$. Differentiating
 we get $\frac{d}{dt}(f \circ \gamma) = 0$, so $Df(\gamma(t_0)) \cdot \dot{\gamma}(t_0)$
 $= \nabla f(p) \cdot \dot{\gamma}(t_0) = 0$

$$\langle \nabla f(p), \dot{\gamma}(t_0) \rangle = 0 \quad \square$$

An application of the Implicit funcⁿ thm:
 $\rightarrow$

4-5 Generalized level sets : We'll discuss level sets for $f: U \subseteq \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$, for smooth f and define the tangent space $T_p S$ of $S = f^{-1}(0)$ at $p \in S$ as the set of all vectors $v \in \mathbb{R}^{n+k}$ such that $v = \dot{\gamma}(0)$ for a smooth $\gamma: I \rightarrow S$, $\gamma(0) = p$, $0 \in I$. We sketch a proof of

Theorem : Let $f: U \subseteq \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ be C^1 . Assume $S := f^{-1}(0) \neq \emptyset$ and $\forall p \in S$, $Df(p)$ has rank k . Then $T_p(S) = \text{Ker } Df(p)$. In particular $T_p(S)$ is an n -dim'l vector space $/\mathbb{R}$.

Proof (Sketch) : - Just like the case $k=1$,

$T_p(S) \subseteq \text{Ker}(Df(p))$ (Chain rule).

• Conversely, let $w \in \mathbb{R}^{n+k}$ be such that $Df(p)(w) = 0$. Since $Df(p)$ has rank k , by composing f with a suitable permutation, we may assume $\left(\frac{\partial f_i}{\partial x_{n+j}} \right)_{1 \leq i, j \leq k}$ is invertible.

$\leadsto$

4th b. Write any $z \in U \subseteq \mathbb{R}^{n+k}$ as $z = (x, y)$,
 $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$; $p = (a, b)$, $\omega = (u, v)$.

Let $G: U \subseteq \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ be defined by

$$G(x, y) := (x, f(x, y)). \text{ Then}$$

- $DG(p)$ is invertible (?). So $\exists$
nbds U_1 of p in $\mathbb{R}^{n+k}$ and V of a in $\mathbb{R}^n$
and a C^1 map $h: V \rightarrow \mathbb{R}^k$ such that

- $h(a) = b$; $\{(x, y) \in U_1 \mid f(x, y) = 0\}$

$$= \{(x, h(x)) \mid x \in V\}.$$

- Consider $\gamma(t) := a + tu$ ($\omega = (u, v)$).

$\Rightarrow \gamma(t) \in V$ for sufficiently small t .

- $C(t) := G^{-1}((\gamma(t), 0))$. Then C is

$$C^1 \text{ as } G \text{ is; } C(0) = \bar{G}^{-1}((\gamma(0), 0)) \\ = \bar{G}^{-1}((a, 0)) = (a, b) = p.$$

- $\dot{C}(0) = \omega$. This follows from the
computation $DG(p)(\omega = (u, v)) = (u, 0)$.



4-7. Given $f: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ as in the theorem and $S = f^{-1}(0)$ the corresponding level set, for $p \in S$, $Df(p): \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ (rank k) $\Rightarrow \ker(Df(p)) = T_p(S)$.

When $k=1$, we have $T_p(S) = \nabla f(p)^\perp \subseteq \mathbb{R}^{n+1}$ and $\nabla f(p)$ is the normal to the hypersurface S , $\forall p \in S$.

Towards a differentiable structure on S :

Let $S = f^{-1}(0)$, $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ a C^1 -map with $\nabla f(p) \neq 0 \forall p \in S$ (the rank condⁿ).

Fix $p_0 = (a_1, \dots, a_{n+1}) \in S$, so $\nabla f(p_0) \neq 0$.

Let $Q_0 = (a_1, \dots, a_n) \in \mathbb{R}^n$ be the projⁿ of p_0 onto $\mathbb{R}^n$. By the Implicit funcⁿ thm, $\exists$

an open set $V_0 \ni Q_0$, an interval $(a_{n+1} - \delta, a_{n+1} + \delta)$ and a C^1 -map g on V_0

such that $f(x_1, \dots, x_n, g(x_1, \dots, x_n)) = 0$

in $V_0 \quad \rightsquigarrow$

4-8. $a_{n+1} = g(a_1, \dots, a_n)$, $|a_{n+1} - g(a_1, \dots, a_n)| < \delta$
in V_0 .

• Any solⁿ $(x_1, \dots, x_{n+1}) \in V_0 \times (a_{n+1} - \delta, a_{n+1} + \delta)$
of the eqⁿ $f(x_1, \dots, x_{n+1}) = 0$ is of
the form $x_{n+1} = g(x_1, \dots, x_n)$.

Let $U_0 = S \cap V_0 \times (a_{n+1} - \delta, a_{n+1} + \delta)$. Then

$p_0 \in U_0$ and the restriction of the
Projection: $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ to U_0 gives

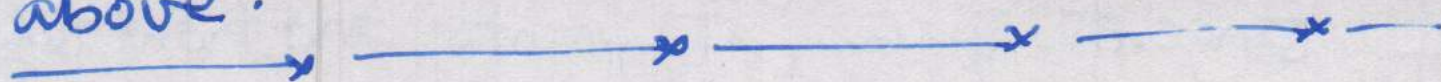
$$\varphi_0: U_0 \rightarrow V_0, \varphi_0(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n) \in V_0,$$

with $\varphi_0^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_n, g(x_1, \dots, x_n))$.

• φ_0 is a diffeom.

Thm: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be smooth, $S = f^{-1}(0) \neq \emptyset$,
the Jacobian of f has rank m at all pts
of U . Then every point of S has an
open nbd diffeomorphic to an open
subset of $\mathbb{R}^{n-m}$.

Proof is an exercise for you. Next
class: Important examples of S as
above.



Lecture-5

Examples :- 1. Let $S = \{X \in M_n(\mathbb{R}) \mid X^t = X\}$. Then

S is a vector space $/\mathbb{R}$ of $\dim n(n+1)/2$.

Let S be identified with $\mathbb{R}^{n(n+1)/2}$ and

$f: \mathbb{R}^{n^2} \equiv M_n(\mathbb{R}) \rightarrow \mathbb{R}^{n(n+1)/2} \equiv S$ be defined by

$f(X) = XX^t$. We are interested in

$O(n, \mathbb{R}) := f^{-1}(I)$; the orthogonal

group $= \{X \in M_n(\mathbb{R}) \mid XX^t = I\}$. We'll

check the rank condition on f . We have

$$Df(A)(X) = \lim_{t \rightarrow 0} [f(A+tX) - f(A)]$$

$$\lim_{t \rightarrow 0} \frac{1}{t} [(A+tX)(A+tX)^t - AA^t] = AX^t + XA^t. \quad \text{for } A \in O(n)$$

Claim: $Df(A)$ has rank $n(n+1)/2$; To see this,

we must show that for $Y \in S$, $\exists X \in M_n(\mathbb{R})$ with $AX^t + XA^t = Y = Y^t$. If we find X

with $AX^t = Y/2$, then $Y^t = Y \Rightarrow X$ would give

a solution to $AX^t + XA^t = Y$. Now $AX^t = Y/2$

$\Rightarrow X = YA/2$ is a solution. Hence $Df(A)$

has full rank $\forall A \in O(n)$. By the theorem done in the last class, $O(n)$ is a 'manifold'.

⁵⁻² of dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

We also note the computation: $DF(I)(x) = x + x^t$,

$$\text{So } \text{Ker}(DF(I)) = \{x \in M_n(\mathbb{R}) \mid x + x^t = 0\} \\ = \text{Skew}_n(\mathbb{R}) = \{x \in M_n \mid x^t = -x\}.$$

Hence $T_I(O(n)) = \text{Skew}_n(\mathbb{R})$, the tangent space at I to $O(n)$; $\mathfrak{o}(n) := T_I(O(n))$.

2. Consider $f: M_n(\mathbb{R}) \rightarrow \mathbb{R}$, $f(A) = \det(A)$.

Then $f^{-1}(1) =: SL_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) \mid \det x = 1\}$.

For $X = (x_{ij})_{1 \leq i, j \leq n}$, $\det X$ is a polynomial in x_{ij} 's, hence $f \in C^\infty$ (?). We will compute $\nabla f(A)$ for $A \in SL_n(\mathbb{R})$. We note

$$\text{that } \det X = x_{11} \det X_{11} - x_{12} \det X_{12} + \dots \\ + (-1)^{n+1} x_{1n} \det X_{1n}$$

Where X_{1j} is the submatrix of X obtained by deleting the 1st row & j th column.

In particular, X_{1j} involves variables except those in the 1st row, i.e. x_{1k} 's. Hence

$$\frac{\partial f}{\partial x_{11}} = \det X_{11}, \text{ and therefore}$$

$$\frac{\partial f}{\partial x_{11}}(I) = 1 \neq 0.$$

s-3

Let $A_0 \in SL_n(\mathbb{R})$ be arbitrary. Define

$y_{ij} = ij^{\text{th}}$ entry of $A_0^{-1}X$. So, for $X=A_0$

we get $Y = (y_{ij}) = I$ and $f(X)$

$$= f(A_0 Y) = \det(A_0 Y) = \det(A_0) f(Y).$$

$$\text{We have therefore } \frac{\partial f}{\partial y_{11}}(I) = \sum_{i,j} \frac{\partial f}{\partial x_{ij}}(A_0) \frac{\partial x_{ij}}{\partial y_{11}}.$$

Hence for some (i,j) , we must have

$$\frac{\partial f}{\partial x_{ij}}(A_0) \frac{\partial x_{ij}}{\partial y_{11}} \neq 0 \text{ and so } \frac{\partial f}{\partial x_{ij}}(A_0) \neq 0.$$

This shows $\nabla f(A_0) \neq 0$ and thus $SL_n(\mathbb{R})$ is a manifold of $\dim n^2 - 1$, i.e. every matrix in $SL_n(\mathbb{R})$ has an open nbhd $\simeq \mathbb{R}^{n^2-1}$.

→ To compute $T_I(SL_n(\mathbb{R}))$, we must

compute $\ker(Df(I))$. We first define:

Multilinear maps: A bi-linear map $B: V \times W \rightarrow Z$, V, W, Z vector spaces / a field K is a map such that it is linear in each variable:

$$\bullet B(a_1 v_1 + a_2 v_2, w) = a_1 B(v_1, w) + a_2 B(v_2, w) \rightarrow$$

5-4

- $B(v, b_1 w_1 + b_2 w_2) = b_1 B(v, w_1) + b_2 B(v, w_2)$.

More generally we define an r -linear map $B: V_1 \times \dots \times V_r \rightarrow W$, linear in each variable.

Lemma:- Let $f: V_1 \times \dots \times V_r \rightarrow W$ be a multi-linear map of Euclidean spaces (so $V_i = \mathbb{R}^{n_i}$, $W = \mathbb{R}^m$ etc). Then f is differentiable with

$$Df(v_1, v_2, \dots, v_r)(x_1, \dots, x_r) = \sum_{i=1}^r f(v_1, \dots, v_{i-1}, x_i, v_{i+1}, \dots, v_r) \quad (?)$$

Proof:- We have $Df(v_1, \dots, v_r)(x_1, \dots, x_r)$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left[f(v_1 + tx_1, \dots, v_r + tx_r) - f(v_1, \dots, v_r) \right]$$

We first expand $\uparrow$ in $\uparrow$ powers of t

~~$$= \lim_{t \rightarrow 0} \frac{1}{t} \left[f(tx_1, v_2, \dots, v_r) + f(v_1, tx_2, v_3, \dots, v_r) + \dots \right]$$~~

$$= f(v_1, v_2, \dots, v_r) + f(tx_1, v_2, \dots, v_r) +$$

$$f(v_1, tx_2, v_3, \dots, v_r) + \dots + f(v_1, \dots, v_{r-1}, tx_r)$$

+ terms involving higher ~~power~~ number of coordinates with t . Hence we have (?)

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left[tf(x_1, v_2, \dots, v_r) + tf(v_1, x_2, v_3, \dots, v_r) + \dots + tf(v_1, \dots, v_{r-1}, x_r) \right]$$

$$s-s = \sum_i f(v_1, \dots, x_i, \dots, v_r). \quad \square$$

Back to $SL_n(\mathbb{R})$:- We think of $M_n(\mathbb{R})$ as $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_n$, vectors in $\mathbb{R}^n$ written column-wise, so $(e_1, \dots, e_n) = I$. We apply the lemma to $\det: M_n(\mathbb{R}) = \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$.

We then have

$$\begin{aligned} & D(\det(I)) (x_1, \dots, x_n) \\ &= \sum_{i=1}^n \det(e_1, \dots, x_i, \dots, e_n). \quad \text{We see (?)} \end{aligned}$$

$$\text{that } \det(e_1, \dots, x_i, \dots, e_n) = \det \begin{pmatrix} 1 & 0 & \dots & x_{1i} & \dots & 0 \\ 0 & 1 & \dots & x_{2i} & \dots & 0 \\ \vdots & 0 & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{ni} & \dots & 1 \end{pmatrix} = x_{ii}.$$

Therefore $D \det(I)(x) = \text{trace}(x)$.

We thus get $\overline{T_I}(SL_n(\mathbb{R})) = \text{Ker}(x \mapsto \text{trace}(x))$
 $= \mathfrak{sl}_n(\mathbb{R}) = n \times n \text{ trace 0 matrices } / \mathbb{R}.$

Exercises: 1. Compute $D \det(A)(x)$ for.

$$\det: GL_n(\mathbb{R}) \rightarrow \mathbb{R}$$

$$2. \text{ Let } U(n) = \{x \in GL_n(\mathbb{C}) \mid x \bar{x}^t = I\}$$

5-6 Where $X^* := X \bar{X}^t$, $\bar{X} = (\bar{x}_{ij})$. Consider $f: \mathbb{C}^{n^2} \rightarrow H$ be the map $X \mapsto XX^*$, Where $H = \{X \in M_n(\mathbb{C}) \mid X = X^*\}$, the $\mathbb{R}$ -vector space of $n \times n$ hermitian = self-adjoint matrices.

(i) Compute $\dim_{\mathbb{R}} H$. (ii) Realize $U(n)$ as $f^{-1}(I)$, Check appropriate conditions on f required for $U(n)$ to be a level set for f as in the case of $O(n)$. (iii) Compute $T_I(U(n))$ as the set $u(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$; Compute $\dim U(n)$ as a real manifold. (iv) Compute the same for $SU(n) = \{X \in U(n) \mid \det X = 1\}$ and show $T_I(SU(n)) = \mathfrak{su}(n) = \{X \in u(n) \mid \text{trace}(X) = 0\}$.

The above discussion describes what are called the 'classical' Lie groups and their 'Lie algebras' as a vector space = the tangent space at the identity element. We'll discuss these in general later.

5-7

Remark 1. The groups SL_n , SO_n , SU_n are respectively called the special linear, special orthogonal and the special unitary group, While GL_n , O_n , U_n are called (resp.) the general linear, orthogonal and the unitary group.

2. Class C^ω : $f: U \rightarrow \mathbb{R}$ be C^∞ . We call f

as real analytic in a nbd of $p \in U$ if $\exists$ $r > 0$ such that the series

$$\sum_{i_1 \geq 0, \dots, i_n \geq 0} \frac{1}{i_1! \dots i_n!} \left[\frac{\partial^{i_1} \partial^{i_2} \dots \partial^{i_n} f}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_n^{i_n}} (p) \right] (x_1 - p_1)^{i_1} \dots (x_n - p_n)^{i_n}$$

converges absolutely to $f(x_1, \dots, x_n)$ for

$\|x - p\| < r$; $p = (p_1, \dots, p_n)$, $x = (x_1, \dots, x_n)$.

The above series is the Taylor series of f at p .

- A function that is real analytic at all $p \in U$ is called a real analytic or C^ω function on U . A map $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^ω if each coordinate function of f is such.

- A function may well be C^∞ (i.e. smooth) yet not C^ω , e.g. $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto e^{-1/x^2}$, $x \neq 0$
 $0 \mapsto 0$ (?). $\leadsto$

5-8
Taylor's theorem:- Let $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ be C^k ,
 $p \in U$ and assume for $x \in U$, the line segment
 $p + tx$, $0 \leq t \leq 1$ lies in U . Let $x^{(r)} = (\underbrace{x, \dots, x}_r)$
and $f^{(r)}(p) x^{(r)} := D^r f(p) (x, \dots, x)$; here
 $D^i f = D(D^{i-1} f)$; $D^{i-1} f: U \rightarrow \mathbb{R}^m$, $p \mapsto (D^{i-1} f)(p)$.

Then

$$f(p+x) = f(p) + \frac{f'(p)x}{1!} + \dots + \frac{1}{k!} f^{(k)}(p) x^{(k)} + \varepsilon_k(x),$$

Where $\frac{\varepsilon_k(x)}{\|x\|^k} \rightarrow 0$ as $x \rightarrow 0$.

Example:- Let $\exp: M_n(\mathbb{R}) \xrightarrow{\sim} GL_n(\mathbb{R})$ be
the map $\exp(x) = e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Let us
discuss this a bit.

The exponential map:- For $X = (x_{ij}) \in M_n(\mathbb{R})$
 $= \text{End}_{\mathbb{R}}(\mathbb{R}^n)$, let $\|X\|_{\infty} = \max_{i,j} |x_{ij}|$. This is a
norm on $M_n(\mathbb{R})$, equivalent to the operator norm
on $\mathbb{R}^n$. Then (Exercise)

- $\|AB\| \leq \|A\| \|B\|$, $\|A^r\| \leq \|A\|^r$
- A sequence $X_k \rightarrow X$ in the operator norm if
and only if $x_{ij}^{(k)} \rightarrow x_{ij} \forall 1 \leq i, j \leq n$ as $k \rightarrow \infty$.
Here $X_k = (x_{ij}^{(k)})$ & $X = (x_{ij}) \leadsto$

- 5-9
- If $\sum_{k=0}^{\infty} \|A_k\|$ is convergent, then $\sum_{k=0}^{\infty} A_k$ converges to an element A in $M_n(\mathbb{R})$.
 - For any $X \in M_n(\mathbb{R})$ the series $e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$ is convergent & the sum is denoted by $\exp(X) = e^X$.
 - $X \mapsto e^X$ is a real analytic map on $M_n(\mathbb{R})$.
 - For $A, B \in M_n(\mathbb{R})$ with $AB = BA$, we have

$$e^{A+B} = e^A e^B = e^B e^A = e^{B+A} \quad (\text{we'll discuss this later}).$$
 - For $A, B \in M_n(\mathbb{R})$, $A e^B A^{-1} = e^{ABA^{-1}}$.
 - Fix $X \in M_n(\mathbb{R})$. Let $\gamma: \mathbb{R} \rightarrow GL_n(\mathbb{R})$ (?) be defined by $\gamma(t) = e^{tX}$. Then $\dot{\gamma}(t) = X \cdot \gamma(t)$.
So $\gamma(0) = I$, $\dot{\gamma}(0) = X$.
 - $e^{tX} \cdot e^{-tX} = I \Rightarrow e^{tX}$ is invertible $\forall t$, and all X . So indeed $X \mapsto e^X \in GL_n(\mathbb{R})$.

— We'll discuss more on this. The exponential map provides an important tool in study of Lie groups and their 'Lie algebras'.

— x ————— x ————— x ————— x ————— x —————

Lecture-6

Differentiable / smooth / C^r / C^ω - manifolds :- In DG-1

You saw geometry of curves and surfaces and these were sub-structures of $\mathbb{R}^2$ or $\mathbb{R}^3$ mostly, where we had the coordinates from the ambient Euclidean space available to us in order to do analysis on these structures; these are needed to talk of differentiability, to make sense of 'direction' on such structures, or to talk of normal at a given point. We wish to treat curves/surfaces now in their own right as geometric objects and talk of differentiable functions etc. on them. The idea is to look at these 'locally' and use coordinates from Euclidean spaces suitably to do calculus. We first define:

Defⁿ: (Topological manifold): A manifold of dimension n is a Hausdorff topological space M such that every $p \in M$ has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$.

A local chart is a pair (U, φ) , $U \subseteq M$ open and $\varphi: U \xrightarrow{\sim} V \subseteq \mathbb{R}^n$, a homeomorphism.

- A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of local charts

⁶⁻² such that $\bigcup_{\alpha \in \Lambda} U_\alpha = M$, is called an atlas.
 $(U_\alpha, \varphi_\alpha)$ is also called a coordinate chart
 and for $p \in U_\alpha$, $\varphi_\alpha(p)$ has coordinates in $\mathbb{R}^n$, which are called coordinates of p
 with respect to $(U_\alpha, \varphi_\alpha)$.

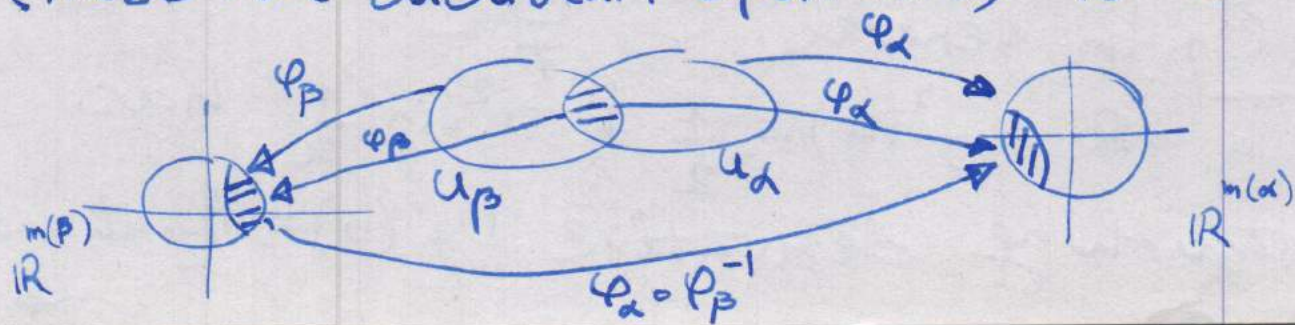
A more general def of a (topological) manifold:
 M Hausdorff, with a covering (open) $\{U_\alpha\}_{\alpha \in \Lambda}$,
 homeomorphisms $\varphi_\alpha: U_\alpha \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^{m(\alpha)}$;
 but then we do not define dimension,

e.g. $M = \underbrace{\quad}_{\mathbb{R}} \sqcup \underbrace{\quad}_{S^2}$

When we wish to talk of dimension of M ,
 we either work with the first defⁿ, or
demand $(M, \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda})$ satisfies the

C^k -compatibility condition; $\exists k \in \mathbb{N} \cup \{\infty\}$
 such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the
 map $\varphi_\alpha \circ \varphi_\beta^{-1}: \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$

(these are Euclidean open sets) is C^k



Remarks: 1. Suppose on a topological manifold M , (U, φ) and (V, ψ) are two charts, $U \cap V \neq \emptyset$. For $x \in U \cap V$, we have two sets of coordinates: the φ -coordinates $\{\varphi_1(x), \dots, \varphi_{m(\varphi)}(x)\}$ and the ψ -coordinates $\{\psi_1(x), \dots, \psi_{m(\psi)}(x)\}$. These, in general would be different. The transition homeomorphisms $\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V)$ & $\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$, express the φ coordinates in terms of the ψ -coords and vice-versa. When M is a C^k -manifold, i.e. $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is a C^k -atlas, the transition maps are C^k -diffeomorphisms.

2. If we assume M to be connected, then, by the inverse function theorem, $m(\alpha) = m(\beta) \forall \alpha, \beta \in \Lambda$ (discuss), so we can define the dimension of M as the common value $m = m(\alpha)$, $\alpha \in \Lambda$.

3. We call an atlas $\mathcal{A}$ on M as smooth or C^∞ if it is $C^k \forall k \in \mathbb{N}$.

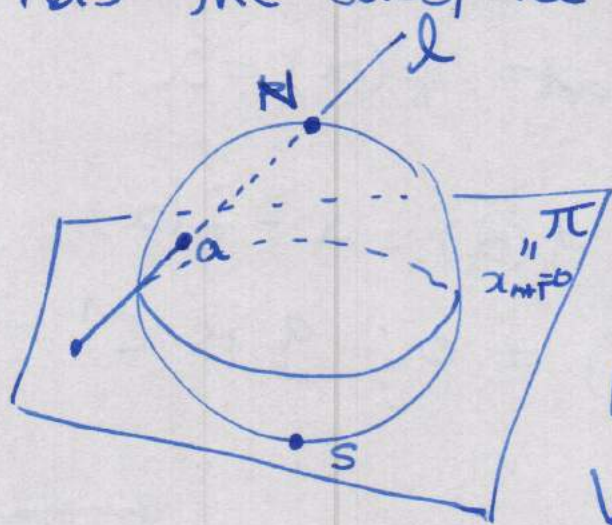
4. We define analytic (or C^ω) manifolds analogously.

6-4
Examples :- 1. Let $U \subset \mathbb{R}^n$ be an open subset, i the inclusion map. Then U is a C^∞ -manifold (in fact C^ω) of dimension n . The C^∞ -atlas $\{(U, i)\}$ does the job in this case.

- Consider $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ as an open subset; so $GL_n(\mathbb{R})$ is an n^2 -dim'l real manifold, of smooth or C^ω type.

2. The n -Sphere S^n : $S^n = \{a \in \mathbb{R}^{n+1} \mid \|a\| = \left(\sum_{i=1}^{n+1} a_i^2\right)^{1/2} = 1\}$

has the subspace topology from $\mathbb{R}^{n+1}$.



Let $N = (0, 0, \dots, 0, 1) \in S^n$ be the 'north pole' and $S = (0, 0, \dots, 0, -1)$ the 'south pole' on S^n .

Let $U = S^n - \{N\}$ and $V = S^n - \{S\}$. Then U, V

are open in S^n , $U \cup V = S^n$. We define the coordinate functions on $U \cup V$ to give a C^∞ -atlas as follows: this is the 'stereographic projection'; for $a \in U$, let l be the line by a and N and π be the plane $\subseteq \mathbb{R}^{n+1}$ given by $x_{n+1} = 0$. Define $\varphi(a)$ = the point of intersection of l and π ; $\varphi: U \rightarrow \mathbb{R}^n = \{x_{n+1} = 0\}$ is given by $\varphi((a_1, \dots, a_{n+1})) = (x_1, \dots, x_n)$, where $x_i = \frac{a_i}{1 - a_{n+1}}$

6-5 for $1 \leq i \leq n$ (note $a_{n+1} \neq 1$ as $a \in U$). Similarly, $\psi: V \rightarrow \mathbb{R}^n$ is given by $a = (a_1, \dots, a_{n+1}) \mapsto (y_1, \dots, y_n)$ where $y_i = \frac{a_i}{1+a_{n+1}}$, $1 \leq i \leq n$.

Check $\psi\psi^{-1}$ & $\psi\psi^{-1}$ are C^∞ . So $(S^n, \{(U, \varphi), (V, \psi)\})$ is a C^∞ -manifold.

(for $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, $(x_1, \dots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} x_i^2 - 1$, $Df(p) \neq 0$ for $p \in S^n = f^{-1}(0)$ and the implicit function thm applies to get a nbd for $p \in S^n$, projecting onto the 'plane' $x_{n+1} = 0$, yielding a C^∞ -atlas).

3. Level sets of smooth functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$ as we discussed, for well behaved functions, are C^k -manifolds. (Example 2.1.11 of Kumaresan).

4. The real projective space $\mathbb{RP}^n \equiv P^n(\mathbb{R})$:- We

let $\mathbb{RP}^n =$ Set of lines through 0 in $\mathbb{R}^{n+1} =$ Set of 1-dim'l subspaces of $\mathbb{R}^{n+1}$. Given a 1-dim'l subspace $\subseteq \mathbb{R}^{n+1}$, it is the set of all $\mathbb{R}$ -multiples of a fixed non zero vector. Any two non zero vectors in a line differ by a scalar $\in \mathbb{R}$. To topologize $\mathbb{RP}^n$ we proceed as follows: On $\mathbb{R}^{n+1} \setminus \{0\}$, define the equivalence $v \sim w \Leftrightarrow v = \lambda w$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

$\leadsto$

6-6 The an equivalence class $[v]$ can be identified with the line $\mathbb{R} \cdot v$; give $\mathbb{RP}^n \equiv \mathbb{R}^{n+1} \setminus \{0\} / \sim$ the quotient topology,

i.e. $U \subseteq \mathbb{RP}^n$ is declared open if its "pullback" is open in $\mathbb{R}^{n+1} \setminus \{0\}$ (hence in $\mathbb{R}^{n+1}$) via the class map $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n, x \mapsto [x]$.

• π is an open map: to see this, for $t \in \mathbb{R} \setminus \{0\}$, define $f_t: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^{n+1} \setminus \{0\}, f_t(x) = tx$. Then f_t is a homeomorphism. Let $U' \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be open. Then $\pi(U') = \bigcup_{t \neq 0} f_t(U')$. Each $f_t(U')$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$, hence $\pi(U')$ is open in $\mathbb{RP}^n$. Using this, check that $\mathbb{RP}^n$ is Hausdorff. Define $U_i, 1 \leq i \leq n+1$, by

$$U_i := \left\{ [x] \in \mathbb{P}^n \mid x_i \neq 0, x = (x_1, \dots, x_i, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\} \right\}.$$

• Defⁿ of U_i makes sense, since for $\lambda \in \mathbb{R} \setminus \{0\}$ and a representative x of $[x] \in U_i$, we have $\lambda x = (\lambda x_1, \dots, \lambda x_i, \dots, \lambda x_{n+1})$ and $\lambda x_i \neq 0$.

• U_i are open in $\mathbb{P}^n$. Define $\varphi_i: U_i \rightarrow \mathbb{R}^n$ by $\varphi_i([y]) = \left(\frac{y_1}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_{n+1}}{y_i} \right) \rightsquigarrow$

6-7 Where $(x_1, \dots, x_{n+1})$ is an arbitrary elem. of $[y]$. Then φ_i is well defined (?).

- φ_i is continuous, 1-1 and onto (?).

- $\varphi_i \circ \varphi_j^{-1}$ is C^∞ , where defined, for $1 \leq i, j \leq n+1$. Hence $\mathbb{R}P^n = \mathbb{P}_{\mathbb{R}}^n = \mathbb{P}^n(\mathbb{R})$ is a smooth manifold of dim n . ($\bigcup_i U_i = \mathbb{P}^n$).

Exercise :- $\mathbb{R}P^n \cong \frac{S^n}{\{\pm 1\}}$; $x, y \in S^n, x \sim y \Leftrightarrow x = \pm y$.

Example 5 : Grassmann manifolds : These generalize the construction of $\mathbb{P}^n$. Fix $k, 1 \leq k \leq n$. Let $G(k, n)$ denote the set of all k -dim'l vector subspaces of $\mathbb{R}^n$. So $\mathbb{R}P^n = G(1, n+1)$. We proceed to give a C^∞ manifold structure on $G(k, n)$. Let $F(k, n)$ denote the set of all k -frames in $\mathbb{R}^n$; a k -frame in $\mathbb{R}^n$ is (an ordered) set x of k -lin. indep. vectors in $\mathbb{R}^n$,

$x_1 = (x_{11}, \dots, x_{1n}), x_2 = (x_{21}, \dots, x_{2n}), \dots,$

$x_k = (x_{k1}, \dots, x_{kn})$. We think of x as

a $k \times n$ matrix with rows $x_1, \dots, x_k$.

$\rightarrow M_{k \times n}(\mathbb{R}) \equiv \mathbb{R}^{kn}$, is a smooth manifold.

$F(k, n)$ is the set of rank k -matrices. $\rightarrow$

6-8
→ $F(k, n)$ is an open subset of $\mathbb{R}^{kn}$:

$X \in M_{k \times n}(\mathbb{R})$ has rank $k \Leftrightarrow$ rows are lin. indep.
 $\Leftrightarrow$ not all $k \times k$ minor determinants are zero. (supply details).

Hence $F(k, n)$ is a smooth manifold.

→ We have the natural map $F(k, n) \xrightarrow{\pi} G(k, n)$
 $X \mapsto$ subspace spanned by $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$

→ $X = (x_1, \dots, x_k)$ & $Y = (y_1, \dots, y_k)$ give the same subspace $\subseteq \mathbb{R}^n \Leftrightarrow Y = CX$ for a non singular $\equiv$ invertible $k \times k$ matrix C .

(Compare with $\mathbb{R}P^n$ construction).

→ Define $\sim$ on $F(k, n)$ by $Y \sim X$ iff $\exists C \in GL_k(\mathbb{R})$ with $Y = CX$. Then $G(k, n) = F(k, n) / \sim$; $\pi: F(k, n) \rightarrow F(k, n) / \sim$

is the map we defined above. Give $G(k, n)$ the quotient topology from $F(k, n)$.

→ $[G(1, \mathbb{R}) = \mathbb{R}^* = \mathbb{R} \setminus \{0\}]$.

→ π is an open map (Exercise).

C^∞ -compatible charts: We'll use $k \times k$ submatrices of $X \in M_{k \times n}(\mathbb{R})$ for this.

6.9

Let $J = (j_1, \dots, j_k)$ be an ordered subset of $(1, \dots, n)$ and J' be its complementary subset e.g. $J = (1, 2, \dots, k)$, then $J' = (k+1, \dots, n)$. Let X_J be the $k \times k$ sub-matrix (x_{ij_l}) , $1 \leq i, l \leq k$ of the $k \times n$ matrix X and $x_{J'}$, the complementary $k \times (n-k)$ submatrix obtained by deleting out the columns $j_1, \dots, j_k$ of X . Let $\tilde{U}_J = \{X \in F(k, n) \mid X_J \text{ is nonsingular}\}$ & $U_J = \pi(\tilde{U}_J) \subseteq G(k, n)$. Then $\tilde{U}_J$ is open in $F(k, n)$, hence U_J is open in $G(k, n)$.

→ Each $\gamma \in \tilde{U}_J$ is equivalent to exactly one $k \times n$ matrix X in which $X_J = k \times k$ identity matrix e.g. for $J = (1, 2, \dots, k)$, $X = \begin{pmatrix} 1 & \dots & 0 & x_{1,k+1} & \dots & x_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & x_{k,k+1} & \dots & x_{kn} \end{pmatrix}$

• If $X \sim Y$ and X_J is non-singular, then $X = X_J^{-1} Y$. Define $\varphi_J: U_J \rightarrow M_{k \times (n-k)}(\mathbb{R}) \cong \mathbb{R}^{k(n-k)}$ by deleting the k columns corresp to J in the above representative X of Y ; $\varphi_J([Y]) = X_{J'}$.

→ φ_J is well defined, is a homeomorphism onto $\mathbb{R}^{k(n-k)}$; $G(k, n)$ is covered by U_J , $J \subseteq (1, 2, \dots, n)$ $|J| = k$ & $\{(U_J, \varphi_J)\}$ is a C^∞ -atlas on $G(k, n)$.

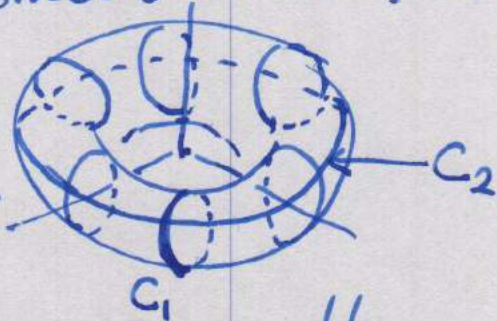
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Lecture-7

Examples (contd) :- Let M, N be C^r -manifolds of dimensions m & n respectively. If $\mathcal{M}$ & $\mathcal{N}$ be atlases on M, N resp. Then $M \times N$ with the product topology (?) and the atlas $\{(U \times V, \varphi_U \times \psi_V) \mid (U, \varphi_U) \in \mathcal{M}, (V, \psi_V) \in \mathcal{N}\}$ gives $M \times N$ a C^r -manifold structure, $\dim M \times N = m + n$.

• The above example gives a smooth manifold structure on $T^2 := S^1 \times S^1$,

Where the first factor can be taken as $S^1 \equiv C_1$ and the second $S^1 \equiv C_2$, so T^2 is obtained by rotating C_1 about the Z -axis. More generally $T^n = S^1 \times \dots \times S^1$ (n factors) is a smooth manifold.



Exercise :- Prove that the Grassmann manifold $G(k, n)$ as defined in Lecture-6, is compact;

[• $O(n, \mathbb{R})$ is compact & $O(n, \mathbb{R}) \xrightarrow{\phi} G(k, n)$
 $\phi([c_1 \dots c_n]) \mapsto \langle c_1, c_2, \dots, c_k \rangle$, c_i columns of $A \in O(n, \mathbb{R})$, $\uparrow$ subspace of $\mathbb{R}^n$ spanned by $c_1, \dots, c_k$
 is continuous & surjective.]

Differential structure on a manifold :- Given a manifold M with two C^r -atlases $\mathcal{A}_1$ & $\mathcal{A}_2$, we wish to ensure the notion of smoothness of a function on M is independent of the choice

⁷⁻² of $\mathcal{A}_i$. To be precise say $f: M \rightarrow \mathbb{R}^m$ and we wish to discuss smoothness of f at $p \in M$. Let $(U, \varphi) \in \mathcal{A}_1$ be a chart, $p \in U$, $\varphi: U \xrightarrow{\sim} \mathbb{R}^n$. We would like to call f smooth if $f \circ \varphi^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth, for all $(U, \varphi) \in \mathcal{A}_1$ with $p \in U$. It may well happen that f is smooth with respect to $\mathcal{A}_1$, but not wrt $\mathcal{A}_2$ (example?). To remove this ambiguity, we

define two C^r -atlases $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ & $\mathcal{B} = \{(V_\beta, \psi_\beta)\}$ to be compatible if for $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$ & $(V_\beta, \psi_\beta) \in \mathcal{B}$, $U_\alpha \cap V_\beta \neq \emptyset$, then $\varphi_\alpha \circ \psi_\beta^{-1}: \psi_\beta(U_\alpha \cap V_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap V_\beta)$ is C^r .

- If $\mathcal{A}$ & $\mathcal{B}$ are compatible, then a function f on M is C^r wrt $\mathcal{A}$ iff f is C^r wrt $\mathcal{B}$.
 - If $\mathcal{A}, \mathcal{B}$ be C^r -atlases on M that are compatible, then the union $\mathcal{A} \cup \mathcal{B}$ is a C^r -atlas.
 - Given a C^r -atlas $\mathcal{A}$, the atlas $\overline{\mathcal{A}} := \bigcup \mathcal{A}'$, union over atlases $\mathcal{A}'$ compatible $\mathcal{A}' \sim \mathcal{A}$ with $\mathcal{A}$, is maximal, containing $\mathcal{A}$.
- Defⁿ: A C^r -maximal atlas on M is called $\rightarrow$

7-3 a differential structure on M . Hence we may think of a diff. structure on M as an equiv. class of atlases with respect to compatibility (C^r). Note that while a topological manifold M has a unique C^0 structure, M may admit several different C^r -differential structures.

Example: $M = \mathbb{R}$; $\mathcal{A} = \{(\mathbb{R}, 1_{\mathbb{R}})\}$ is a C^∞ -manifold, so is $M = \mathbb{R}$ with $\mathcal{B} = \{(\mathbb{R}, \varphi), \varphi(x) = x^3\}$

• $\mathcal{A}$ and $\mathcal{B}$ are not equivalent.

Exercise: The atlases $\{(\mathbb{R}, \varphi_i)\}_{i=0,1,2,\dots}$ are inequivalent smooth atlases on $\mathbb{R}$;
 $\varphi_i: \mathbb{R} \rightarrow \mathbb{R}$, $\varphi_i(x) = x^{2i+1}$.

Point set topology recalls: • We are aiming to do geometry on manifolds and the least we would want is put a metric on them, so we demand these be Hausdorff (metric spaces are such).

• A space X is locally compact if every point has an open nbd whose closure is compact. Manifolds are locally compact, as well as locally path connected, i.e. at every point, there is a local basis of path conn. open sets. $\rightarrow$

Exercise :- A connected topological manifold is path-connected.

Paracompact spaces :- These generalize compact spaces.

Let (X, τ) be a top. space and $\{U_\alpha\}$ be an open covering of X . An open covering $\{V_i\}_{i \in I}$ of X is called a refinement of $\{U_\alpha\}$ if every set V_i is contained in some U_α .

A covering $\{V_i\}_{i \in I}$ is locally finite if every $x \in X$ has a nbd that intersects only finitely many V_i 's. A top space X is paracompact if every open cover of X has a locally finite refinement. (see Munkres for details).

We need paracompactness property for manifolds because that ensures existence of 'partition of unity', a gadget that allows us to construct global objects from local data (we'll see this in detail, one example is the construction of a Riemannian metric).

Second countable space :- We call a top space (X, τ) second countable if it admits a countable basis.

- Let M be a top space that is Hausdorff and locally Euclidean (i.e. locally $\cong \mathbb{R}^n$) and connected. Then the following are equivalent.

7-5 (i) M is second countable. (ii) M is paracompact. More generally, dropping connected assumption on M , the following are equiv.

(i) M is 2nd countable. (ii) M is paracompact and has countably many conn. components.

Finally: (this does what we need/want!)

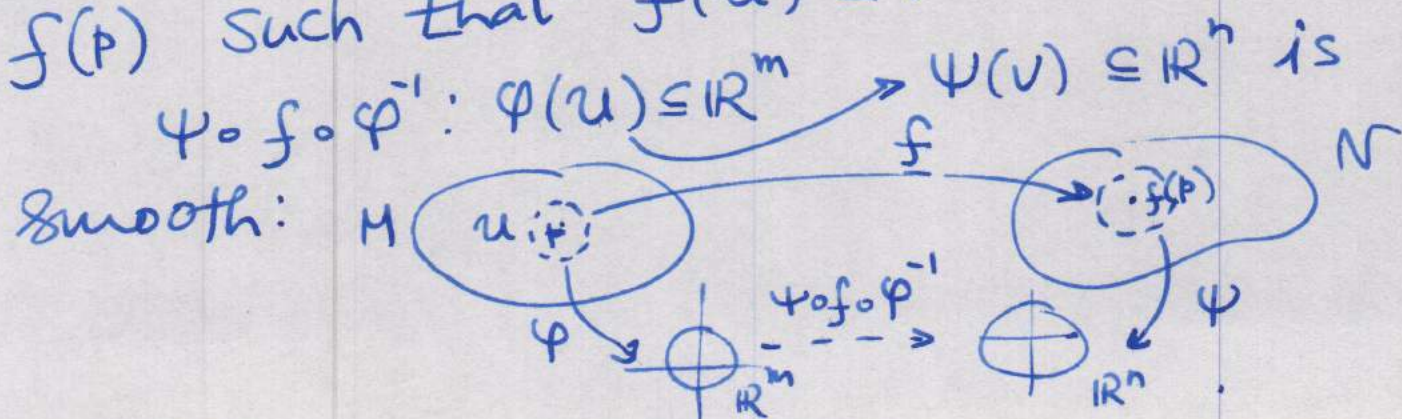
M (Hausdorff), (locally compact and second countable) then M is paracompact. (Munkres)

Defⁿ: A smooth manifold M of dimension n is a Hausdorff topological space satisfying

- M is 2nd countable, paracompact together with a C^∞ -differential $\equiv$ differentiable structure (i.e. equiv. class of a C^∞ -atlas).

We'll work with this defⁿ now on. We now define:

Smooth maps between manifolds: Let M, N be smooth manifolds, $U \subseteq M$ open. A map $f: U \rightarrow N$ is smooth if $\forall p \in U$, $\exists$ coordinates charts (u, φ) around p and (v, ψ) around $f(p)$ such that $f(u) \subseteq v$ and the map



7-6.

- $M \xrightarrow{f} N \xrightarrow{g} S$ smooth $\Rightarrow g \circ f: M \rightarrow S$ is smooth.

- M_1, M_2 smooth $\Rightarrow$ projections $M_1 \times M_2 \xrightarrow{\pi_i} M_i$, $i=1,2$, are smooth.

Algebra of smooth functions :- Let M be a smooth manifold. Let $C^\infty(M) = \{f: M \rightarrow \mathbb{R} \mid f \text{ is smooth}\}$. For $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M)$, define $\alpha f + \beta g \in C^\infty(M)$ as usual and fg by $(fg)(x) = f(x)g(x)$. Then $fg \in C^\infty(M)$. Hence $C^\infty(M)$ is an $\mathbb{R}$ -algebra with these operations. More generally for $U \subseteq M$ open, we have $C^\infty(U)$ as the set of smooth functions on U .

- $C^\infty(U)$ is an $\mathbb{R}$ -algebra, containing constants.

- $V, U \subseteq M$ open, $V \subseteq U$ then $f \in C^\infty(U) \Rightarrow f \in C^\infty(V)$.

- If $U = \bigcup_i U_i$, $U_i \subseteq M$ open and $f: U \rightarrow \mathbb{R}$ be such that $f|_{U_i} \in C^\infty(U_i) \forall i$, then $f \in C^\infty(U)$.

- $\exists$ an integer $m \geq 0$ such that $\forall p \in M$, $\exists$ an open set $U \ni p$ and functions (distinct) $x_1, \dots, x_m \in C^\infty(U)$ such that

(i) the map $\varphi: q \mapsto (x_1(q), \dots, x_m(q))$ is a homeomorphism $U \cong \mathbb{R}^m$ $W \subseteq \mathbb{R}^m$ open.

(ii) $\forall$ open $V \subseteq U$ & $f: V \rightarrow \mathbb{R}$, $f \in C^\infty(V) \Leftrightarrow f \circ \varphi^{-1} \in C^\infty(\varphi(V))$.

7-7. The proofs are left to you. The above is a very useful way to define a smooth manifold.

Example: Lie groups :- A Lie group is a group G with a smooth $\equiv$ differentiable manifold structure and we demand that the product $\mu: G \times G \rightarrow G$, $\mu(x, y) = xy$ and the inverse $i: G \rightarrow G$, $x \mapsto x^{-1}$ are both smooth. (equiv. $(x, y) \mapsto xy^{-1}$ is smooth).

• $(\mathbb{R}^n, +)$, $(\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \cdot)$, $(S^1, \cdot)$ are Lie groups.

• $GL_n(\mathbb{R})$, $O(n, \mathbb{R})$, $U(n)$, $SO(n)$, $SU(n)$, $SL(n, \mathbb{R})$ are Lie groups.

Exercise: G be a Lie group and $a, b, g \in G$.

Let $R_a, L_b, A_g: G \rightarrow G$ be the maps $R_a(x) = xa$, $L_b(x) = bx$, $A_g(x) = g x g^{-1}$.

These are smooth maps, hence (?) are diffeomorphisms; $L_a \equiv$ left translation by a , $R_a \equiv$ right translation, $A_g \equiv$ conjugation by g .

Having defined the notion of smooth = diff. functions between manifolds, we would like to discuss derivatives of such. $\rightarrow$

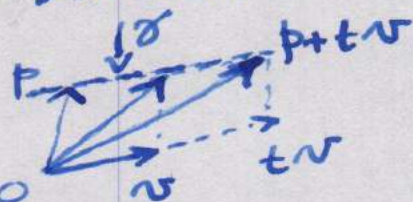
7-8 Going by the case of (open subsets of) $\mathbb{R}^n$, the derivative at a point should be a linear map, but the domain and codomain are not clear at this stage! We need to first define Tangent space :- Philosophically speaking, since a manifold does not come with an embedding into a Euclidean space, we cannot use the techniques we used in case of (open subsets / curves, surfaces) $\mathbb{R}^n$. This will be defined 'intrinsically', using the diff. structure at hand. In this, we'll see several 'avatars' of the tangent space, all equally useful and we'll have natural isomorphisms between them, allowing us to pass from one description to the other. First let us discuss these ideas for open subsets of $\mathbb{R}^n$.

Directional derivative: Let $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $v \in \mathbb{R}^n$. The directional derivative of f at p in the direction of v is the limit $D_v f(p) := \lim_{t \rightarrow 0} \frac{1}{t} [f(p+tv) - f(p)]$, if it exists. Let $g(t) := f(p+tv) \in \mathbb{R}^m$.

7-9 Then for $I \ni 0$, an open interval $\subseteq \mathbb{R}$, we

$$(*) \quad D_{\vec{v}} f(p) = \left. \frac{d}{dt} g(t) \right|_{t=0} = Df(p)(\vec{v}).$$

If $\gamma(t) = p + t\vec{v}$, then γ is a linear curve with $\gamma(0) = p$ & $\dot{\gamma}(0) = \vec{v}$.



The point here is that for a small ball $\ni p$ and small values of t , $\gamma(t)$ lies in the ball. So, to evaluate the limit $(*)$ above, we restrict f to the curve $\gamma(t)$. We could have used any curve for this:

Lemma :- Let $\sigma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^n$ be a smooth curve with $\sigma(t) \in U \quad \forall t \in (-\epsilon, \epsilon)$; $\sigma(0) = p$ and $\dot{\sigma}(0) = \vec{v}$, then

$$D_{\vec{v}} f(p) = \left. \frac{d}{dt} (f \circ \sigma(t)) \right|_{t=0}.$$

Proof :- By Chain rule we have

$$\begin{aligned} \left. \frac{d}{dt} (f \circ \sigma(t)) \right|_{t=0} &= Dg(p)(\vec{v}) \\ &= Df(p)(\dot{\sigma}(0)) \\ &= Df(p)(\vec{v}) = D_{\vec{v}} f(p). \end{aligned}$$

□



7-10 Hence we can think of the domain of $Df(p)$ as the set of tangent vectors to curves passing through p and lying in U . We have shown that $Df(p)$ is independent of the choice of a curve γ in U with $\gamma(0)=p$, $\dot{\gamma}(0)=v$.

Let $U \subseteq \mathbb{R}^n$ be open, $\gamma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^n$ be a curve lying in U , $\gamma(0)=p$. Let $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$. Then the tangent to γ at p is $\dot{\gamma}(0) = (\dot{\gamma}_1(0), \dots, \dot{\gamma}_n(0))$. Let $v \in \mathbb{R}^n$ be any vector. Since U is open, $p \in U$, $\exists \delta > 0$ s.t. $B(p, \delta) \subseteq U$ and hence the linear curve $\gamma = \gamma_{p,v}(t) = p + tv$ as above, lies in U & has $\gamma(0)=p$, $\dot{\gamma}(0)=v$. Hence the set $T_p(U)$ of all tangent vectors at p can be identified with $\mathbb{R}^n$.

We may regard two curves σ, γ through p , lying in U , as equivalent if $\dot{\sigma}(0) = \dot{\gamma}(0)$, i.e. σ and γ have the same tangent vector at p .

(1) Show that this equiv is indep. of choice of $U \ni p$, i.e. $T_p(U) = T_p(\Omega)$ for any open $\Omega \ni p$.

(2) The above indeed is an equiv. rel.

Lecture-8

In the last lecture we started a discussion on tangent spaces and saw that for $U \subseteq \mathbb{R}^n$ open and $p \in U$, $T_p(U) = \mathbb{R}^n$, where we

set $T_p(U) = \text{set of equiv. classes of smooth curves in } U \text{ under the equiv. rel}^n \sigma \sim \gamma \text{ if } \sigma(0) = \gamma(0) = p, \dot{\sigma}(0) = \dot{\gamma}(0)$. [curves are 1-dim'l mfw's].

If we write $[\gamma]$ for the equiv. class of γ for γ as above, then $T_p(U) = \{[\gamma] \mid \gamma \subseteq U\}$.

Tangent vectors as directional derivative operators

We work with $U \subseteq \mathbb{R}^n$ as above. Let $v \in \mathbb{R}^n (= T_p(U), p \in U)$ and $f \in C^\infty(U)$. Then, for $x \in U$, we have the directional derivative $D_v f(x)$, of f at x , in the direction of v ;

$$D_v f(x) = \lim_{t \rightarrow 0} [f(x + tv) - f(x)]$$

$$= \langle \text{grad } f(x), v \rangle.$$

$$\text{For } p \in U, D_v f(p) = \langle \nabla f(p), v \rangle = \sum_i v_i \frac{\partial f}{\partial x_i} \Big|_p$$

$$=: \sum_i v_i \frac{\partial}{\partial x_i} \Big|_p (f) \quad \rightarrow$$

⁸⁻² where, in the last expression we wish to think of $f \mapsto \sum_i v_i \frac{\partial f}{\partial x_i} \Big|_p$ as an operator on $C^\infty(U)$. We have not used before that to compute $D_v f(p)$, we use any curve γ in U , $\gamma(0)=p$ and $\dot{\gamma}(0)=v$, without affecting the value of $\lim_{t \rightarrow 0} \frac{1}{t} (f(p+t\gamma) - f(p)) = \frac{d}{dt} (f \circ \gamma(t)) \Big|_{t=0}$. (see Lemma, Lecture 7).

We can therefore associate the operator $f \mapsto D_v(f)(p) =: D_{p,v}(f)$ to the equiv. class $[\gamma]$, $\gamma(0)=p$, $\dot{\gamma}(0)=v$.

- $D_{p,v}(f) = D_{p,w}(f) \quad \forall f \in C^\infty(U), U \ni p; \quad v=(v_1, \dots, v_n), w=(w_1, \dots, w_n)$

$$\Rightarrow D_{p,v}(x_i) = v_i = D_{p,w}(x_i) = w_i, \quad 1 \leq i \leq n.$$

Hence $v=w$. So the map $\mathbb{R}^n \rightarrow \{D_{p,v} \mid v \in \mathbb{R}^n\}$
 $v \mapsto D_{p,v}$

is injective.

- The set $\{D_{p,v} \mid v \in \mathbb{R}^n\}$ is a vect sp/ $\mathbb{R}$:

$$(\alpha D_{p,v} + \beta D_{p,w})(f) := \alpha \cdot D_{p,v}(f) + \beta \cdot D_{p,w}(f),$$

$\alpha, \beta \in \mathbb{R}$. Hence $v \mapsto D_{p,v}$ is an isom. of $\mathbb{R}^n$

8-3 $\mathbb{R}$ -vector spaces. We note that the ops $D_{p,v}$ have the following properties:

(i) $D_{p,v}(f) = D_{p,v}(g)$ for f, g smooth in a nbd $\ni p$ (possibly different nbds) and $f = g$ on a nbd $\ni p$.

• This is easy, since the nbds of defⁿ/smoothness for f & g intersect $\neq \emptyset$, say in W , then $D_{p,v}(f)$ & $D_{p,v}(g)$ can be computed, e.g. using curve $\subseteq W$.

(ii) $D_{p,v}(\alpha f + \beta g) = \alpha D_{p,v}(f) + \beta D_{p,v}(g)$,
 $\alpha, \beta \in \mathbb{R}$.

(iii) $D_{p,v}(fg) = f(p)D_{p,v}(g) + D_{p,v}(f)g(p)$

$$\hookrightarrow = \sum_i v_i \frac{\partial}{\partial x_i}(fg)|_p = \sum_i \left[v_i \left(\frac{\partial f}{\partial x_i} \Big|_p \right) g(p) + f(p) \frac{\partial}{\partial x_i}(g)|_p \right] \text{ etc.}$$

Lemma (Characterization of directional derivative operators): Let $C^\infty(p) = \mathbb{R}$ -vector space of $\mathbb{R}$ -valued functions, smooth in a nbd of p (the nbd may depend on a function). Then for an $\mathbb{R}$ -linear map $\delta: C^\infty(p) \rightarrow \mathbb{R}$ satisfying (iii) above: $\delta(fg) = f(p)\delta(g) + \delta(f)g(p) \leadsto$

8-4 $\forall f, g \in C^\infty(p)$ (this is an $\mathbb{R}$ -algebra (?)),
then $\exists! v \in \mathbb{R}^n$ such that $\delta = D_{p,v}$.

Proof! - We'll sketch this (see Pati/Kumarasan).

- For constant functions $c \in C^\infty(p)$, $\delta(c) = 0$ (follows from the product formula).

- May assume $p = 0 = \text{origin in } \mathbb{R}^n$ (?).

- Taylor formula for f around 0 $\Rightarrow$

$$(*) \quad f(x) = f(0) + \sum_i x_i g_i(x), \quad x = (x_1, \dots, x_n)$$

in a small ball around 0 & g_i 's are smooth on this ball, $g_i(0) = \frac{\partial f}{\partial x_i} \big|_0$.

- by (*),
$$\begin{aligned} \delta(f) &= \delta\left(f(0) + \sum_i x_i g_i(x)\right) \\ &= \sum_i \delta(x_i) g_i(0) \quad \left[\text{given by product formula for } \delta\right] \\ &= \left(\sum_i \delta(x_i) \frac{\partial}{\partial x_i} \big|_0 \right) (f). \end{aligned}$$

- $v := (\delta(x_1), \dots, \delta(x_n)) \Rightarrow \delta = D_{p=0, v}$. $\square$

Point derivations :- Given an $\mathbb{R}$ -algebra A ,
a derivation, with values in $\mathbb{R}$, is an
 $\mathbb{R}$ -linear map $\delta: A \rightarrow \mathbb{R}$ such that -
 $\delta(ab) = \delta(a)b + a\delta(b)$, $a, b \in A$.

8-5 More generally, when A is a commutative ring and M an A -module, we define $\delta: A \rightarrow M$ as above. The directional derivative operators, by the lemma, are precisely the 'point derivations' on $C^\infty(p)$, i.e. those satisfying the product formula $\delta(fg) = f(p)\delta(g) + \delta(f)g(p)$. Hence we have

$$\begin{aligned} T_p(\mathbb{R}^n) &\equiv T_p(U) = \left\{ [\gamma] \mid \gamma \subseteq U \text{ smooth, } \gamma(0) = p \right\} \frac{d}{dt} \bigg|_{t=0} \\ &= \left\{ \delta \mid \delta \text{ a point derivation on } C^\infty(p) \right\}. \end{aligned}$$

Remarks: (i) In finding/solving for ' v ' for a given point derivation δ , we need in (*) that f be smooth; if f were only C^1 e.g. then the g_i 's in (*) may be only continuous and then δ cannot be applied.

(ii) Given $f: U \rightarrow \mathbb{R}^m$, $p \in U$, $f = (f_1, \dots, f_m)$,

$f_i = \alpha_i \circ f$, we have

$Df(p): \mathbb{R}^n \rightarrow \mathbb{R}^m$. We may now

think of $\mathbb{R}^n = T_p(U)$, $\mathbb{R}^m = T_{f(p)}(\mathbb{R}^m)$

3-6 and identify $Df(p)(v) = (Df_1(p)(v), \dots, Df_m(p)(v))^t$ as a column vector and hence $Df(p)(v)$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}|_p & \dots & \frac{\partial f_1}{\partial x_n}|_p \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}|_p & \dots & \frac{\partial f_m}{\partial x_n}|_p \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Tangent space to a manifold: - Let now M be a smooth manifold, $p \in M$. Let $C^\infty(p) =$ Set of smooth $\mathbb{R}$ -valued functions defined in a nbd of p ; i.e. $f: U_f \rightarrow \mathbb{R}$ is smooth on the open set $U_f \subseteq M$.

Def: - A tangent vector v at $p \in M$ is a point derivation of $C^\infty(p)$, i.e. $v: C^\infty(p) \rightarrow \mathbb{R}$ is $\mathbb{R}$ -linear and $v(fg) = f(p)v(g) + v(f)g(p)$, $f, g \in C^\infty(p)$.

We denote the $\mathbb{R}$ -vector space of all tangent vectors to M at p by $T_p(M)$.

Some elements of $T_p(M)$: - Let (U, φ) be a coordinate chart around p and $\varphi(p) = 0 \in \mathbb{R}^n$. Let x_i ($1 \leq i \leq n$) denote the coordinate functions on U and u_i be the coordinates

γ^{-1} in $\mathbb{R}^n$. Then, for $f \in C^\infty(p)$, define

$$\begin{aligned}\frac{\partial}{\partial x_i} \Big|_p (f) &:= \frac{\partial}{\partial u_i} (f \circ \varphi^{-1}) (0) \\ &= \frac{\partial}{\partial u_i} \Big|_0 (f \circ \varphi^{-1}).\end{aligned}$$

Then $\frac{\partial}{\partial x_i} \Big|_p \in T_p(M)$ (check this, easy!)

• When $f = x_i$, $\frac{\partial}{\partial x_i} \Big|_p (f) = \frac{\partial}{\partial u_i} (\underbrace{x_i \circ \varphi^{-1}}_{u_i}) (0)$
 $= \frac{\partial}{\partial u_i} (u_i) (0) = 1.$

Derivatives of curves on M :- Let $\gamma: (-\epsilon, \epsilon) \rightarrow M$ be a smooth curve through p , i.e. $\gamma(0) = p$.

Define $\dot{\gamma}(0)$ by the point derivation

$$\dot{\gamma}(0)(f) := \frac{d}{dt} (f \circ \gamma)(0) =: \frac{d}{dt} \Big|_{t=0} (f \circ \gamma).$$

Then

• $\dot{\gamma}(0)(f+g) = \dot{\gamma}(0)(f) + \dot{\gamma}(0)(g)$

• $\dot{\gamma}(0)(fg) = \frac{d}{dt} ((fg) \circ \gamma)(0)$
 $= f(p) (\dot{\gamma}(0)(g)) + (\dot{\gamma}(0)(f)) g(p)$

as is easily checked (see Kumarisan).

Hence $\dot{\gamma}(0) \in T_p(M)$. We have $\leadsto$

8-8

- Theorem :- Let M be a smooth manifold, $p \in M$ and (U, φ) be a chart around p , x_i the local coordinates on U , $1 \leq i \leq m = \dim M$.
- Let $v \in T_p(M)$. Then $v = \sum_i v(x_i) \frac{\partial}{\partial x_i} \Big|_p$
 - $\left\{ \frac{\partial}{\partial x_i} \Big|_p, 1 \leq i \leq m \right\}$ are lin. indep./ $\mathbb{R}$.
 - $\dim T_p M = m = \dim M$.

Proof :- Assume wlg that $\varphi(p) = 0$. Then $\varphi(U) \subseteq \mathbb{R}^m$ is an open set $\ni 0$ and $\bar{f} := f \circ \varphi^{-1} \in C^\infty(0)$.

Then the Taylor formula for $\bar{f}$ gives

$$(\#) \quad \bar{f}(x) = \bar{f}(0) + \sum_i u_i G_i(x), \text{ where}$$

$$G_i \text{ are smooth around } 0 \text{ \& } G_i(0) = \frac{\partial}{\partial u_i} \bar{f}(0).$$

Composing $(\#)$ by φ yields

$$(*) \quad f(x) = f(p) + \sum x_i g_i(x);$$

$$x_i = u_i \circ \varphi, \quad g_i = G_i \circ \varphi.$$

$$\text{Hence } v(f) = v(f(p)) + \sum_i [v(x_i) g_i(p) + x_i(p) v(g_i)].$$

We have $g_i(p)$

$$= G_i \circ \varphi(p) = G_i(0) = \frac{\partial}{\partial u_i} \Big|_0 (f) = \frac{\partial}{\partial u_i} \Big|_0 (f \circ \varphi^{-1}).$$

We defined $\frac{\partial}{\partial x_i} \Big|_p (f) := \frac{\partial}{\partial u_i} \Big|_0 (f \circ \varphi^{-1})$. Hence

$$g_i(p) = \frac{\partial}{\partial x_i} \Big|_p (f).$$



8-9. But $\nabla(f(p)) = 0 = \alpha_i(p)$ as $\varphi(p) = 0$. Hence

$$\nabla = \sum_i \nabla(\alpha_i) \frac{\partial}{\partial \alpha_i} \Big|_p.$$

[discuss the case of φ when $\varphi(p) \neq 0$]. $\square$

Linear independence of $\frac{\partial}{\partial \alpha_i} \Big|_p$: We have

already computed $\frac{\partial}{\partial \alpha_i} (\alpha_i) = 1$. We have

$$\frac{\partial}{\partial \alpha_i} \Big|_p (\alpha_j) = \frac{\partial}{\partial u_i} (\alpha_j \circ \varphi^{-1})(0) = \frac{\partial}{\partial u_i} (u_j)(0)$$

$$= \delta_{ij}. \text{ Hence } \left\{ \frac{\partial}{\partial \alpha_i} \Big|_p, 1 \leq i \leq m \right\} \text{ form}$$

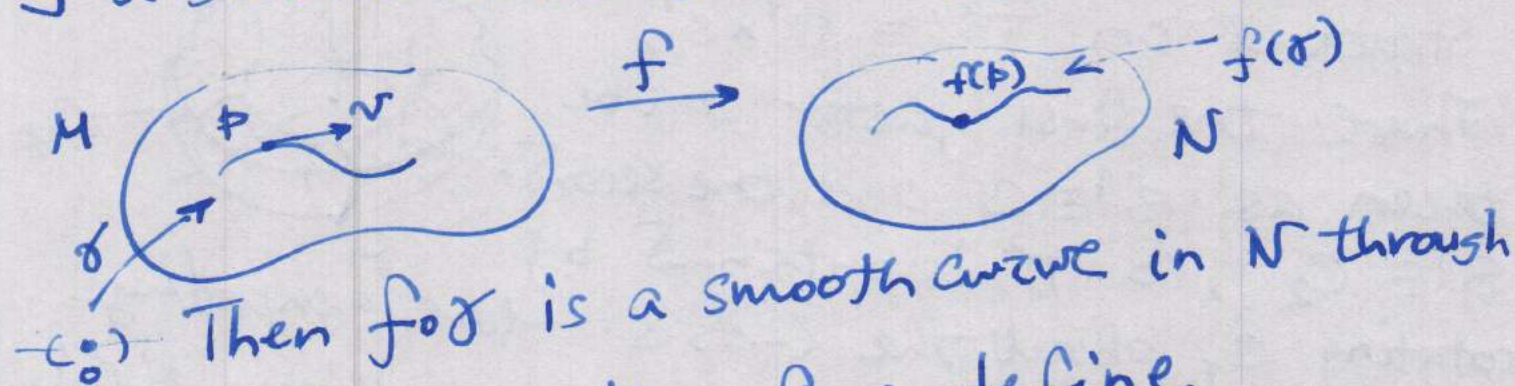
a basis of $T_p(M)$ and thus $\dim_{\mathbb{R}} T_p(M) = m$.

*Lemma: For any $v \in T_p(M)$, $\exists$ a smooth curve γ on M , $\gamma(0) = p$ & $\dot{\gamma}(0) = v$.

Proof: Class discussion exercise. $\square$

This Lemma allows us to identify $T_p(M)$ with the set of equivalence classes of smooth curves γ on M with $\gamma(0) = p$; under the equiv. $\gamma \sim \sigma$ if $\gamma(0) = \sigma(0) = p$ & $\dot{\gamma}(0) = \dot{\sigma}(0)$ as elements of $\text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R}) = \text{point derivations on } C^\infty(p)$.

8-10 Derivatives of smooth maps: Let M, N be smooth manifolds of $\dim m, n$ resp, $f: M \rightarrow N$ be a smooth map. In analogy with the case of $M = U \subseteq \mathbb{R}^m$, $N = \mathbb{R}^n$, we would like the derivative of f at $p \in M$ to be a linear map $Df(p): T_p M \rightarrow T_{f(p)} N$. We use the earlier Lemma for this. Let $v \in T_p M$. Then $\exists$ a smooth curve γ on M , $\gamma(0) = p$, $\dot{\gamma}(0) = v$.



$\therefore$ Then $f \circ \gamma$ is a smooth curve in N through $f(p)$. We can therefore define $Df(p)(v) := (f \circ \gamma)'(0)$. Since $f \circ \gamma$ is a smooth curve through $f(p)$ on N , $(f \circ \gamma)'(0) \in T_{f(p)} N$.

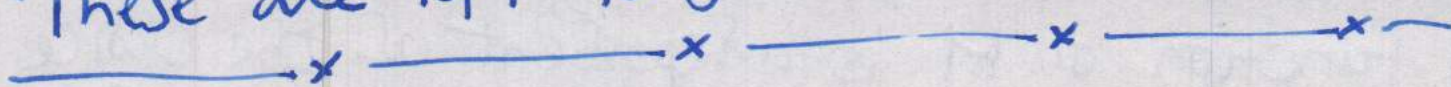
- $Df(p)(v)$ is well defined.

- $Df(p): T_p M \rightarrow T_{f(p)} N$ is a linear

map.

- $Df(p)(v)(g) = v(g \circ f)$, $g \in C^\infty(f(p))$.

These are left to you as exercises.



Lecture-9

Remarks :- In the last lecture we developed the notion of tangent vector to a smooth manifold at a point and the notion of the derivative of a smooth map between two such manifolds. In this, we dealt with the $\mathbb{R}$ -algebra $C^\infty(p)$ of smooth $\mathbb{R}$ -valued functions, smooth in a nbd of $p \in M$. We may think of $C_p^\infty(p) = \{ (f, U_f) \mid f \text{ smooth on } U_f, \text{ open nbd of } p \}$. The relⁿ

$(f, U_f) \sim (g, U_g)$ if $f = g$ on $U_f \cap U_g$ is an equiv. relⁿ on $C^\infty(p)$. The class $[(f, U_f)]$ is called germ of f at p . The set of germs of smooth functions $= \mathcal{O}_p$ (M understood) is an $\mathbb{R}$ -algebra: $[(f, U_f)] + [(g, U_g)] := [(f+g, U_f \cap U_g)]$, $[(f, U_f)] \cdot [(g, U_g)] = [(fg, U_f \cap U_g)]$ etc.

• The evaluation at p map $\mathcal{O}_p \xrightarrow{e_p} \mathbb{R}, [(f, U_f)] \mapsto f(p)$ is surjective $\mathbb{R}$ -algebra map (?).

Let $\ker e_p =: \mathfrak{m}_p$. Then $\mathfrak{m}_p$ is a max ideal.

• Exercise: $\mathcal{O}_p$ is a local ring, i.e. has a unique max ideal.

⁹⁻² We have the induced isom. $\mathcal{O}_p \xrightarrow{m_p} \mathbb{R}$.
 mapping $[(f, \varphi_f)] + m_p \mapsto f(p)$.

We have therefore an $\mathcal{O}_p$ -module structure on $\mathbb{R}$; $[(f, \varphi_f)] \cdot r := f(p)r$ (check).

We therefore can view a 'point derivation' on $C^\infty(p)$ as we defined, as a derivation

$\delta: \mathcal{O}_p \rightarrow \mathbb{R}$, $\mathbb{R}$ treated as $\mathcal{O}_p$ -module via the evaluation action. Then

$$\begin{aligned}\delta([f][g]) &= [f] \cdot \delta([g]) + \delta([f]) \cdot [g] \\ &= f(p) \delta([g]) + \delta([f]) g(p).\end{aligned}$$

• Let M be a smooth manifold of dim m .

Given $p \in M$ and $v \in \mathbb{R}^m$, how do we interpret v as a tangent vector $\equiv$ derivation, at p ?
 For this, fix a chart (U, φ) around p , $\{u_1, \dots, u_m\}$ be the Euclidean (usual) coordinates on $\varphi(U)$, $\varphi(p) = 0 \in U$ and $x_i = u_i \circ \varphi$, $1 \leq i \leq m$. Then v corresponds to the derivation $\sum_i v_i \frac{\partial}{\partial x_i} |_p$ on $\mathcal{O}_p$. $v = (v_1, \dots, v_m)$.

• $p \in M \xrightarrow{f} N \supseteq V \xrightarrow{g} \mathbb{R}$, $f(p) \in V \subseteq N$, V open. Then

$$Df(p)(v) \in T_{f(p)} N \quad \& \quad Df(p)(v)(g) = v(g \circ f)$$

$$\forall g \in C_{f(p)}^\infty(N) \text{ or } [g] \in \mathcal{O}_{f(p)}$$



^{g.3} Matrix representation of $Df(p)$:- Let $q = f(p)$,
 (U, φ) be a chart around p , $\varphi(p) = 0 \in \mathbb{R}^m$ & (V, ψ)
 a chart around q , $\psi(q) = 0$. Let $\{x_i\}$ & $\{y_j\}$
 be the coordinates on U & V resp; so

$x_i(z) = u_i(\varphi(z)) \quad \forall z \in U$, $y_j(z) = u_j(\psi(z))$
 $\forall z \in V$, $\{u_i\}_i$ being the Euclidean coords.

Then, as we proved, $\left\{ \frac{\partial}{\partial x_i} \Big|_p \right\}_i$ and

$\left\{ \frac{\partial}{\partial y_j} \Big|_q \right\}_j$ are bases for $T_p(M)$ and

$T_q(N)$ resp; $Df(p): T_p M \rightarrow T_q N$ is a linear

map, we wish to write its matrix wrt the
 above bases. Write $v = \frac{\partial}{\partial x_i} \Big|_p$, $w = Df(p)(v)$.

Then $w = \sum_j w(y_j) \frac{\partial}{\partial y_j} \Big|_q = \sum_j [Df(p)(v)](y_j) \frac{\partial}{\partial y_j} \Big|_q$

$$= \sum_j v(y_j \circ f) \frac{\partial}{\partial y_j} \Big|_q = \sum_j \frac{\partial}{\partial x_i} \Big|_p (y_j \circ f) \frac{\partial}{\partial y_j} \Big|_q$$

Hence the matrix for $Df(p)$ is $\left(\frac{\partial}{\partial x_i} \Big|_p (y_j \circ f) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$

$n =: \dim N$; which is the usual
 Jacobian matrix in the Euclidean set up. $\square$

• Using the formula $D(f)(p)(g) = v(g \circ f)$,
 it is clear that $D(f)(p): T_p M \rightarrow T_{f(p)} N$ is linear.

3-4.
 • For $I: M \rightarrow M$, the identity map, $D(I)(p): T_p M \rightarrow T_p M$ is identity. Since, for $v \in T_p M$
 $\forall g \in C_p^\infty(M)$, $D(I)(p)(v)(g) = v(g \circ I) = v(g) = v(g)$.

Hence $D(I)(p)(v) = v \quad \forall v \in T_p(M)$.

• Chain rule: $M \xrightarrow{f} N \xrightarrow{g} W$ smooth,

then $D(g \circ f)(p) = Dg(f(p)) \circ Df(p)$.

We have, for $\varphi \in C_{(g \circ f(p))}^\infty(W)$ $v \in T_p M$

$$[D(g \circ f)(p)(v)](\varphi) = v(\varphi \circ (g \circ f)) = v((\varphi \circ g) \circ f). \text{ Now}$$

$$\varphi \circ g \in C^\infty(f(p)). \text{ Hence } = Df(p)(v)(\varphi \circ g) = D(g)(f(p))(Df(p)(v))(\varphi). \quad \square$$

Corollary: Let $\varphi: M \rightarrow N$ be a diffeomorphism with inverse $\psi = \varphi^{-1}: N \rightarrow M$. For $p \in M$,
 $D\varphi(p): T_p M \rightarrow T_{\varphi(p)} N$ is bijective and

$$(D\varphi(p))^{-1} = D\psi(\varphi(p)).$$

Proof: Observe: $D\psi(\varphi(p)) = D(\psi \circ \varphi)(p) = D(I)(p) = Id_{T_p M}$

$$D\varphi(p) \circ D\psi(\varphi(p)) = D(\varphi \circ \psi)(\varphi(p)) = D(Id_N)(\varphi(p)) = Id_{T_{\varphi(p)} N}. \quad \square$$

9-5
Exercise :- Let M_1, M_2 be smooth, $\pi_i: M_1 \times M_2 \rightarrow M_i$ be the projections. These are differentiable. Compute $D\pi_i$, $i=1,2$.

Examples :- 1. Let $M = (-\epsilon, \epsilon)$, $\epsilon > 0$ with the usual manifold structure (as open $\subseteq \mathbb{R}$) and N be a smooth manifold. Let $\gamma: M \rightarrow N$ be smooth. Then, for $t \in M$, $D\gamma(t)(\frac{d}{du}|_t) \in T_{\gamma(t)}N$. We have, for $f \in C^\infty(\gamma(t))$,

$$D\gamma(t)(\frac{d}{du}|_t)(f) = \frac{d}{du}|_t(f \circ \gamma(t)).$$

For $p = \gamma(0)$, let (U, φ) be a coordinate chart in N with $\varphi(p) = 0 \in \mathbb{R}^n$; then, by defⁿ of $\dot{\gamma}(0)$ we have, in terms of the coordinates $x_1, \dots, x_n$ on U , $\dot{\gamma}(0) = \sum \frac{d\gamma_i}{du}|_0 \frac{\partial}{\partial x_i}|_p$

where $\gamma_i = x_i \circ \gamma$ in U .

2. In the above we used the chart $u: t \mapsto t$ on $(-\epsilon, \epsilon) \subseteq \mathbb{R}$ and $T_0 M = \mathbb{R} \cdot \frac{d}{du}|_0$.

If $f \in C^\infty(N)$ and $v \in T_p N$, then $Df(p)(v) \in T_{f(p)}(\mathbb{R}) = \mathbb{R} \cdot \frac{d}{du}|_{f(p)}$ and $Df(p)(v) = d(v) \cdot \frac{d}{du}|_{f(p)}$

where $d(v) = v(f) \in \mathbb{R}$. To see this, we

$$\begin{aligned} \text{Compute: } Df(p)(v) &= [Df(p)(v)](u) \cdot \frac{d}{du}|_{f(p)} \\ &= v(u \circ f) \cdot \frac{d}{du}|_{f(p)} = v(f) \cdot \frac{d}{du}|_{f(p)}. \end{aligned}$$

Cotangent space:- Let M be a smooth manifold, $p \in M$. Then $(T_p M)^* = \text{Hom}_{\mathbb{R}}(T_p M, \mathbb{R})$ is called the cotangent space to M at p and elements of $T_p M^*$ are called cotangent vectors.

Let $f \in C^\infty(M)$, so $f: M \rightarrow \mathbb{R}$ is smooth.

Hence $Df(p): T_p M \rightarrow T_{f(p)} \mathbb{R} \equiv \mathbb{R}$ is given by

$$Df(p)(v) = v(f) \frac{d}{du} \Big|_{f(p)} \text{ as we discussed. We}$$

can therefore think of $Df(p)$ as a linear functional on $T_p M$; $v \mapsto v(f)$. We'll denote this by $df(p)$, i.e. $df(p)(v) = v(f)$.

If (U, φ) be a chart around p with coords $x_1, \dots, x_m$; we get $dx_i(p) \in T_p M^*$. Let us compute $dx_i(p) \left(\frac{\partial}{\partial x_j} \Big|_p \right) = \frac{\partial}{\partial x_j} \Big|_p (x_i|_p)$

Hence $dx_1(p), \dots, dx_m(p)$ are lin indep $\mathbb{R}$ and form a basis of $T_p M^*$. By the above, $\{dx_1(p), \dots, dx_m(p)\}$ is the dual basis corresp to $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_m} \Big|_p \right\}$, $dx_i(p) = \left(\frac{\partial}{\partial x_i} \Big|_p \right)^*$. $\leadsto$

g-7
 • For $f \in C^\infty(M)$, we can think of $p \mapsto df(p)$ as the map $df: M \rightarrow \bigcup_p T_p^* M$ which has $df(p) \in T_p^* M$. We can write a local expression for df :

Lemma!- Let $(p \in M \text{ and}) x_1, \dots, x_m$ be local coordinates (around p) on $U \ni p$, then,

$$df(x) = \frac{\partial f}{\partial x_1}(x) dx_1 + \dots + \frac{\partial f}{\partial x_m}(x) dx_m.$$

Proof!- For $p \in U$ and $v \in T_p M$ we have

$$df|_p(v) = df|_p \left(v_1 \frac{\partial f}{\partial x_1}|_p + \dots + v_m \frac{\partial f}{\partial x_m}|_p \right).$$

On the other hand, $dx_i|_p \left(v_1 \frac{\partial}{\partial x_1}|_p + \dots + v_m \frac{\partial}{\partial x_m}|_p \right) = v_i$ by the calculⁿ $dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}$.

$$\text{Hence } df|_p(v) = \frac{\partial f}{\partial x_1}|_p dx_1|_p^{(v)} + \dots + \frac{\partial f}{\partial x_m}|_p^{(v)} dx_m|_p.$$

$$\therefore df|_p = \sum_i \frac{\partial f}{\partial x_i}|_p dx_i|_p. \quad \square$$

Remarks!- 1. For $f \in C^\infty(M)$, df as above is what is called a (smooth) differential 1-form (linear functionals are called linear forms) on M . We'll study these in general soon.

g-8
2. The spaces $\bigcup_p T_p M$ and $\bigcup_p T_p^* M$ carry natural differential structures and hence are smooth manifolds, denoted by $T(M)$ & $T(M)^*$, called the tangent and cotangent bundle of M resp. We'll discuss these.

Submanifolds: Given a manifold M and $S \subseteq M$ we would like to define S as a smooth submanifold if S is a manifold in its own right, with some compatibility conditions.

Before we proceed, we need to study maps $f: M \rightarrow N$ with conditions on Df .

Immersion, submersion: Let $f: M \rightarrow N$ be smooth and $p \in M$. We call f an immersion at p if $Df(p)$ is injective, submersion at p if $Df(p)$ is surjective and f is a local diffeomorphism at p if $Df(p)$ is bijective: $T_p M \rightarrow T_{f(p)} N$.

Call f an immersion if f is an immersion at all $p \in M$, similarly define f to be a submersion.

Example (Canonical immersion/submersion):

• For $m \leq n$, $\iota: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be the map $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$. Clearly this is an immersion, called the canonical immersion.

9-9
If $m \geq n$, the map $\mathbb{R}^m \rightarrow \mathbb{R}^n$ given by
 $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$ is clearly a
submersion, called the canonical submersion.

A non-example :- A smooth injective map
may fail to be an immersion, e.g. $\varphi: \mathbb{R} \rightarrow \mathbb{R}$,
 $\varphi(x) = x^3$ is not one (why?).

Theorem :- 1. Let $f: M \rightarrow N$ be an immersion on a
neighbourhood of $p \in M$. Then $\exists$ coord charts
 $(U, \varphi) \ni p$, $(V, \psi) \ni q = f(p)$ in N such that
 $\psi \circ f \circ \varphi^{-1}$ is a canonical immersion.

2. If $f: M \rightarrow N$ is a submersion at $p \in M$. Then
 $\exists$ charts $(U, \varphi) \ni p$, $(V, \psi) \ni q = f(p)$ such that
 $\psi \circ f \circ \varphi^{-1}$ is a canonical submersion.

Proof: 1. Choose chart $(U, \varphi) \ni p$ such that $x_i(p) = 0$
for $1 \leq i \leq m = \dim M$ and $(V, \psi) \ni q = f(p)$ with coords
 y_i , $y_i(q) = 0$, $1 \leq i \leq n = \dim N$.

• Choose a basis $\{v_1, \dots, v_m\}$ of $T_p M$ and $\{w_i\}_{1 \leq i \leq n}$
of $T_q N$ such that $Df(p)u_i = w_i$ ($u_i \mapsto w_i$)
for $1 \leq i \leq m$. Let $G := \psi \circ f \circ \varphi^{-1}: \varphi(U) \rightarrow \mathbb{R}^n$.

• May assume by modifying basis of $\mathbb{R}^m$, $\mathbb{R}^n$ if nec.
that $DG(\varphi(p))$ is canonical: $\mathbb{R}^m \rightarrow \mathbb{R}^n$. $\rightarrow$

9-10. Let $H: U \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$ be the map $H(x, z) = G(x) + (0, z)$, where $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \ni (0, z)$. Then (?) $DH|_{(p, 0)}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity. $\Rightarrow H$ is a local diffeomorphism at $(p, 0)$. Hence the neighbourhoods $(U, \varphi) \ni p$ and $(V, H \circ \varphi^{-1})$ do the job. (something to check).

2. Here we assume $Df(p)$ is onto: $T_p M \rightarrow T_{f(p)} N$, so has full rank, $m \geq n$. Hence we may assume that the Jacobian matrix of f at p has its first $n \times n$ minor invertible, so

$\left(\frac{\partial f_i}{\partial x_j}(p) \right)_{1 \leq i, j \leq n}$ is invertible. Consider $\eta: V \rightarrow \mathbb{R}^n$

by $\eta(z) = (f_1(z), \dots, f_n(z))$, $z \in V$; f_i being the coordinate functions of f on V .

• Then η is a local diffeom at $q = f(p)$.

The charts $(U, \varphi) \ni p$ & $(W, \eta|_W) \ni q$ do the job, where $\eta|_W$ is a diffeo, $W \subseteq V$.

Next: discuss submanifolds using the above discussion notions, examples etc.

Lecture-10

Exercises: 1. Let S be a level set of a smooth map $f: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$ so that S is a smooth mfd. Prove that the inclusion $\iota: S \rightarrow \mathbb{R}^{n+k}$ is an immersion. So $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an immersion.
 2. The quotient map $\pi: S^n \rightarrow \mathbb{RP}^n$ is an immersion as well as a submersion (See Kum.)

Submanifolds!: Given a smooth manifold, $N \subseteq M$, we would like N to be a smooth submanifold if the inclusion is smooth. Another point is that for a "smooth submanifold" N , $p \in N$, we would like $T_p N$ to be a subspace of $T_p M$. Finally, restrictions of smooth maps $\gamma: M \rightarrow W$ to N should be smooth. These requirements suggest:

Def!: A submanifold of a smooth manifold M is a pair (S, ι) , where S is a smooth manifold, $\iota: S \rightarrow M$ is a 1-1 immersion.

Examples: 1. Level surface manifolds (Exc 1) are submanifolds of the ambient Euclidean space
 2. (U, ι) , $U \subseteq M$ open subset & $\iota = \text{inclusion}$.

Exercise: $f: M \rightarrow N$ be smooth, then f is continuous.

Remarks (see Kum.) 1. If (S, ι) is a submanifold of M , the subspace topology on $\iota(S) \subseteq M$ induces a topology on S , which in general is coarser than the topology S comes with.

2. For $p \in S$, $\exists U$ open $\subseteq S$, $p \in (U, \alpha)$, $q = \iota(p) \in (V, \gamma) \subseteq M$ such that

$$\gamma \circ \bar{\alpha}'(x_1, \dots, x_s) = (y_1, \dots, y_m, 0, \dots, 0);$$

$$\text{i.e. } \gamma \circ \bar{\alpha}'(\alpha(U)) = \{q \in \gamma(V) \mid y_i(q) = 0, i \geq m+1\}.$$

Call (U, α) , (V, γ) as above as adapted charts.

Defⁿ: A smooth map $\varphi: M \rightarrow N$ is an embedding if φ is 1-1 immersion on M . Call φ , an embedding, regular if $\varphi: M \rightarrow \varphi(M)$ is a homeom.

3. If $\varphi: M \rightarrow N$ is a regular embedding, then the original topology on M & the one induced from N via φ coincide.

4. A submanifold of N is a pair (M, ι) with ι an embedding. A regular submanifold is (S, ι) with ι a regular embedding.

Regular values of smooth maps: Let $f: M \rightarrow N$ be a smooth map of manifolds. $\rightarrow$

¹⁰⁻³ We call a point $q \in N$ a regular value of f if f is a submersion at all $p \in \bar{f}^{-1}(q)$.

We then have:

- Propⁿ₁: Let $q \in N$ be a regular value of $f: M \rightarrow N$. Then $(S = \bar{f}^{-1}(q), \iota)$ is a regular submanifold of M , $\iota = \text{inclusion}: S \rightarrow M$,
— $\dim S = \dim M - \dim N$. (for proof 'localize')
- Propⁿ₂: Let $f: M \rightarrow N$ be smooth and $q \in N$ a regular value; $(S = \bar{f}^{-1}(q), \iota)$ the corresp. regular submanifold. Then, for $p \in S$,
 $T_p S$ is a subspace of $T_p M$ and
 $T_p S = \ker Df(p) \subseteq T_p M$.

• Remarks on the Proof of Prop. 2: By Prop. 1, (S, ι) is a regular submanifold of M , hence $D\iota(p): T_p S \rightarrow T_p M$ is 1-1. Hence we may identify $T_p S$ with $D\iota(p)(T_p S)$.

- f is constant on $S \Rightarrow T_p S \subseteq \ker Df(p)$ (details you should supply).

• $Df(p)$ is onto: $T_p M \rightarrow T_p N$ (f is a submersion at p)

$$\therefore \dim \ker Df(p) = \dim M - \dim N = \dim T_p S.$$



¹⁰⁻⁴
Vector fields :- Let M be a smooth manifold.

By a smooth vector field on M , we mean a map $X: M \rightarrow T(M) = \bigcup_p T_p M$ such that

- $X(p) \in T_p M \quad \forall p \in M$
- For any coordinate chart $U \ni p$ with coordinates $x_1, \dots, x_m$; Write $X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i} \big|_q, q \in U$.

We demand that f_i are smooth with respect to any local chart in the smooth atlas of M .

Remark: When we understand the tangent bundle $T(M)$ is a smooth manifold, we'll see that $X: M \rightarrow T(M)$ with $X(p) \in T_p M$, is smooth if it is so as a map of manifolds.

Example: $U \subseteq \mathbb{R}^n$ be open, $X_i(p) = \frac{\partial}{\partial x_i} \big|_p$ for $p \in U$. Then X_i is a smooth vector field on U , $1 \leq i \leq n$. For $U = \mathbb{R}^n$ and $p \in \mathbb{R}^n$, $X(p) = \sum f_i(p) \frac{\partial}{\partial x_i} \big|_p, f_i \in C^p(\mathbb{R}^n)$ is a smooth vector field.

- Any smooth vect. field on $\mathbb{R}^n$ is as above. $\leadsto$

10-5
Remarks: 1. As we discussed tangent vectors to M at p as derivations on $C^\infty(p)$, we'll see that we can regard smooth vector fields on M as derivations: $C^\infty(M) \rightarrow C^\infty(M)$.

2. Let $k \geq 1$ & $n = 2k-1$, $M = S^n \subset \mathbb{R}^{n+1}$, the n -sphere, n odd. Then we have seen that $T_p S^n \subseteq T_p \mathbb{R}^{n+1}$. For $p \in S^n$, let

$$X(p) = \left[\left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right) + \left(-x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) + \dots + \left(-x_{2k} \frac{\partial}{\partial x_{2k-1}} + x_{2k-1} \frac{\partial}{\partial x_{2k}} \right) \right]_p$$

Exercise: • $X(p) \in T_p(S^n) \forall p \in S^n$, so X is a smooth vector field on S^n .

- $X(p) \neq 0 \forall p \in S^n$.
- For S^n , n even, every (continuous) vector field X admits a zero, i.e. $\exists p \in S^n, X(p) = 0$. (See Kum.)

Vector fields as derivations: Let M be smooth, $C^\infty(M) = \mathbb{R}$ -algebra of smooth functions $M \rightarrow \mathbb{R}$. Let $\mathcal{X}(M) :=$ Set of all smooth vector fields on M .

¹⁰⁻⁶. We have natural operations on $\mathcal{X}(M)$:

$$(X+Y)(p) = X(p) + Y(p) \in T_p M, \quad p \in M;$$

$$\text{For } \alpha \in \mathbb{R}, X \in \mathcal{X}(M), (\alpha \cdot X)(p) = \alpha \cdot X(p) \in T_p(M).$$

Then $\bullet X+Y, \alpha X \in \mathcal{X}(M)$.

This makes $\mathcal{X}(M)$ into an $\mathbb{R}$ -vector space.

$C^\infty(M)$ -module structure on $\mathcal{X}(M)$: Let f

$\in C^\infty(M)$ and $X \in \mathcal{X}(M)$. We define $f \cdot X$

$$\text{by } (f \cdot X)(p) := f(p)X(p) \in T_p M. \text{ Since}$$

$f \in C^\infty(M)$, it follows that $f \cdot X \in \mathcal{X}(M)$.

Hence $\mathcal{X}(M)$ is a $C^\infty(M)$ -module.

Derivations of $C^\infty(M)$: Let $X \in \mathcal{X}(M)$.

Define $\tilde{X}: C^\infty(M) \rightarrow C^\infty(M)$ by $\tilde{X}(f)(p) := X(p)(f)$.

Then $\bullet \tilde{X}(f) \in C^\infty(M), \tilde{X}(\alpha f + \beta g) = \alpha \tilde{X}(f) + \beta \tilde{X}(g)$.

$$\bullet \tilde{X}(fg) = \tilde{X}(f) \cdot g + f \cdot \tilde{X}(g).$$

$\rightarrow$ We'll prove that any derivation of $C^\infty(M)$ comes from a smooth vector field.

Functions (smooth) with compact support:-

$$\bullet \text{ Let } f: \mathbb{R} \rightarrow \mathbb{R} \text{ be } f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then $f \in C^\infty(\mathbb{R})$ (but not analytic at 0!). $\rightarrow$

10-7. Let $0 < a < b$ and define $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ by $\varphi(x) = \frac{f(b^2 - \|x\|^2)}{[f(b^2 - \|x\|^2) + f(\|x\|^2 - a^2)]}$.

• Then $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(x) = \begin{cases} 1 & \text{if } x \in B(0, a) \\ 0 & \text{if } x \notin B(0, b). \end{cases}$
 $\rightarrow$ For $\varphi \in C^\infty(U)$, $U \subseteq \mathbb{R}^n$ open, we define

Support of $\varphi = \overline{\{x \in U \mid \varphi(x) \neq 0\}}$.

Prop: $K \subseteq \mathbb{R}^n$ be compact & $U \subseteq \mathbb{R}^n$ be open, $K \subseteq U$. Then $\exists f \in C^\infty(\mathbb{R}^n)$, $f \equiv 1$ on K , $f = 0$ on $\mathbb{R}^n \setminus U$.

Prf!: $C = \mathbb{R}^n \setminus U$ is closed, $C \cap K = \emptyset$.

Let $2\epsilon := d(K, C) = \inf\{d(x, y) \mid x \in K, y \in C\}$.

$\rightarrow \epsilon > 0$. Let $x_1, \dots, x_k \in K$ be such that

$K \subseteq \bigcup_i B(x_i, \epsilon/2)$. Choose $\psi_j \in C^\infty(\mathbb{R}^n)$, $\psi_j = 1$

on $B(x_j, \epsilon/2)$, $\psi_j = 0$ for $x \notin B(x_j, \epsilon)$. Then

$\varphi(x) := 1 - \prod_{j=1}^k (1 - \psi_j)$ has the req.

property. ~~✗~~

Exercise:- Let $U \subseteq \mathbb{R}^n$ open, $f \in C^\infty(U)$, $x_0 \in U$. Then $\exists V \subseteq U$ open, $x_0 \in V$ & $g \in C^\infty(\mathbb{R}^n)$ s.t. $\begin{cases} f(x), & x \in V \\ g(x) = 0, & x \notin U. \end{cases}$

¹⁰⁻⁸
Exercise 2 :- We construct smooth functions on chart nbds in M by using such functions on (open subsets of) $\mathbb{R}^m$ and pulling them back by the corresp. charts.

• To get smooth maps $: M \rightarrow \mathbb{R}$, let (U, φ) be a coord. chart on M , $F: \varphi(U) \rightarrow \mathbb{R}$ be smooth with compact support $\subseteq \varphi(U)$. Define $f: M \rightarrow \mathbb{R}$ by $f = F \circ \varphi$ on U and $f = 0$ on $M \setminus U$. Then $f \in C^\infty(M)$ has compact support $\subseteq U$.

Theorem :- Let $\delta: C^\infty(M) \rightarrow C^\infty(M)$ be derivation. Then $\delta = \tilde{X}$ for a smooth vector field X on M .

Proof :- 1. Let $X, Y \in \mathcal{X}(M)$ be such that $\tilde{X} = \tilde{Y}$. Then $X = Y$. Suffices to show:

if $\tilde{X}(f) = 0 \forall f \in C^\infty(M)$ then $X = 0$.

We'll show $X(p) = 0 \forall p \in M$. Let (U, φ) be a chart around p . Let $g \in C^\infty(M)$ be such that $g = 1$ in a nbhd of $p \subseteq U$ and $g = 0$ on $M \setminus U$. Let $f := \begin{cases} g x_i & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}$ & coordinates on U . Then

$f \in C^\infty(M)$. By hypothesis $\tilde{X}(f) = 0$. ~>

v^{-9} But $\underbrace{\tilde{X}(f)}_0(p) = X(p)(f) = X(p)(\sum g_i a_i)$
 $= X(p)(a_i) \forall p \in V.$

Hence the local coordinates of $X(p)$ on V are 0. and thus $X(p) = 0 \forall p \in V$. In Part. $X(p) = 0$.

2. If $D: C^\infty(M) \rightarrow C^\infty(M)$ is a derivation and $f = 0$ on an open set $U \subseteq M$, then $Df = 0$ on U . To see this, let $p \in V_p \subset U_p \subset U$, all open, $g \in C^\infty(M)$ be s.t. $g = 1$ on V_p , $g = 0$ on $M \setminus U_p$. Set $h = 1 - g$. Then $f = fh$ on M : if $z \in U_p$, then $f(z) = 0 = h(z)f(z) = (fh)(z)$.

If $z \notin U_p$, $h(z) = (1 - g)(z) = 1$. Hence $f(z) = f(z)h(z) = (fh)(z)$.

• $D(f) = D(fh) = f \cdot D(h) + D(f) \cdot h$. Hence
 $D(f)(p) = \underbrace{f(p)}_0 (Dh)(p) + (Df)(p) \underbrace{h(p)}_0 = 0$.

3. To define a vector field X , we must give $X(p) \in T_p M \forall p \in M$, i.e. a derivation: $C^\infty(p) \rightarrow \mathbb{R}$, $\forall p$. Let $f \in C^\infty(p)$ and define $X(p)(f) := \delta(f)(p)$ for any

¹⁰⁻¹⁰ $g \in C^\infty(M)$, $g=f$ in a nbd of p . Then δ is a derivation $\Rightarrow X(p) \in T_p(M)$.

If $h \in C^\infty(M)$ with $h=f$ in a nbd of p , then $h=g$ in a nbd of p , so $\delta(g-h)=0$ on that nbd; hence $\delta(g)=\delta(h)$. Thus $X(p)$ is well defined. Thus we have a vector field $X: p \mapsto X(p)$ on M .

To show X is smooth, fix a chart $(U, \varphi) \ni p$. There are $f_i \in C^\infty(M)$ s.t. $f_i = x_i$ on a nbd of p . Then the coefficients of X are given by $X(x_i) = X(f_i)$ on U . Since $X(f_i) = \delta(f_i)$ are smooth, it follows that X is smooth. $\square$

Remark ! We have therefore established a natural isomorphism of vector spaces

$$\mathcal{X}(M) \longrightarrow \text{Der}(C^\infty(M), C^\infty(M))$$

$$X \longmapsto \tilde{X}.$$



Lecture-11

Vector fields on manifolds (contd): Recall that for M smooth, a smooth vector field on M is a map $X: M \rightarrow TM = \bigcup_P T_P M$, $X(p) \in T_p M$, $p \in M$ and for any coordinate chart U on M with coordinates $x_1, \dots, x_m$, writing $X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i} \Big|_q$ for any $q \in U$, the functions f_i are smooth with respect to any local chart in the smooth atlas of M . It is illustrative to discuss these in the C^r -category, since some properties of vector field are more visible:

• For a C^{k+1} -manifold M , $k \geq 1$, we talk of C^k -vector fields. For any chart (U, φ) on M , the map $p \mapsto \frac{\partial}{\partial x_i} \Big|_p$, $p \in U$, $1 \leq i \leq \dim M = m$ is a C^k -vector field on U . Where we may define this notion by any of the:

(i) $X: M \rightarrow TM$ is C^k .

(ii) For any chart (U, φ) and $f_1, \dots, f_m$ the unique functions on U defined by $X|_U = \sum_i f_i \frac{\partial}{\partial x_i}$. Then f_i is

$C^k \forall i$.
(iii) $\forall$ open $U \subseteq M$ & $f \in C^k(U)$ we have $\rightarrow$

¹¹⁻² $X(f) \in C^{k-1}(U)$. Where, recall that, for
 $p \in U$, $X(f)(p) := X(p)(f)$. (*)

We discuss the derivational properties of smooth vector fields on smooth manifolds and prove that any derivation $\delta: C^\infty(M) \rightarrow C^\infty(M)$ comes from a unique smooth vector field X on M , i.e. $\hat{X} = \delta$.

• Given $X, Y \in \mathcal{X}(M)$, we can compose $X \circ Y$ and the following reasons compel us to consider $X \circ Y - Y \circ X =: [X, Y]$.

Exercise 1: Let A be a ring and $\delta_1, \delta_2: A \rightarrow A$ be derivations (i.e. additive & satisfy the product rule). Then, in general, $\delta_1 \circ \delta_2$ is not a derivation.

2. $[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \text{Der}(A)$.

Even when M is smooth & $X, Y \in \mathcal{X}(M)$, there is, in general, no vector field Z on M with $Z(f) = X(Y(f)) \forall f \in C^\infty(M)$.

Example: Let $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$ be vector fields on $M = \mathbb{R}^3$.

11-3. Then, for any $f \in C^\infty(\mathbb{R}^3)$,

$$X(Yf) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} + x \frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial z} + x \frac{\partial^2 f}{\partial x \partial z}$$

Thus $X \circ Y$ may $\left(\frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial z} + x \frac{\partial^2}{\partial x \partial z} \right) (f)$

involve (and $Y \circ X$ too) second order diff. operators, hence for $p \in U$, $(X \circ Y)(p)$ may fail to be in $T_p M$. To rectify this,

we consider $[X, Y](f) := X(Y(f)) - Y(X(f))$.

Then

• If X, Y are C^k -vector fields on C^{k+1} -mfd M , then $[X, Y]$ is C^{k-1} and this is determined (i.e. is!) by $Z(f) = X(Y(f)) - Y(X(f)) \forall f \in C^{k-1}(M)$.

In the above example, $Y(X(f))$

$$= \left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right) \left(\frac{\partial f}{\partial x} \right) = \left(\frac{\partial^2}{\partial y \partial x} + x \frac{\partial^2}{\partial z \partial x} \right) (f)$$

Hence $[X, Y](f) = \frac{\partial}{\partial z}(f)$; $[X, Y] = \frac{\partial}{\partial z} \in T(\mathbb{R}^3)$

Lemma: Let M be smooth, $X, Y \in \mathcal{X}(M)$, $p \in M$, $f \in C^\infty(M)$. Then

$$[X, Y](p)(f) = X(p)(Y(f)) - Y(p)(X(f)).$$

Proof: By defⁿ, $[X, Y](f) \in C^\infty(M)$ and is given by $[X, Y](f)(p) = [X, Y](p)(f)$ and $[X, Y](f) = X(Y(f)) - Y(X(f))$. Also, we defined (identifying $\tilde{X} \equiv X$), $X(f)$ by $X(f)(p) = X(p)(f)$. We thus have

$$(X(Y(f)) - Y(X(f)))(p) = X(p)(Y(f)) - Y(p)(X(f))$$

and, ($X \mapsto \tilde{X}$ being additive),

$$\begin{aligned} [X, Y] &= X \circ Y - Y \circ X \quad \& \quad [X, Y](f)(p) \\ &= [X, Y](p)(f) = (X(Y(f)) - Y(X(f)))(p). \end{aligned}$$

□

Defⁿ: Given $X \in \mathcal{X}(M)$, we denote

$\mathcal{L}_X: C^\infty(M) \rightarrow C^\infty(M)$, $\mathcal{L}_X(f)(p) := X(p)(f)$ and call $\mathcal{L}_X(f)$ the Lie-derivative of f with respect to X .

Remark: Recall that for $f \in C^\infty(M)$

11-5
We defined $df: M \rightarrow \bigcup_p T_p^* M$, $df(p) \in T_p^* M$,
for $v \in T_p M$, $df(p)(v) = v(f)$. We
have, by the above defⁿ, $\mathcal{L}_X(f)(p) = X(p)(f)$.

Hence $\mathcal{L}_X(f)(p) = df(X(p))$, $p \in M$, $f \in C^\infty(M)$.

- $\mathcal{L}_{aX+bY} = a\mathcal{L}_X + b\mathcal{L}_Y$, $a, b \in \mathbb{R}$.

- When $M = \mathbb{R}^n$ and $X(p) = v$, $\mathcal{L}_X(f)(p)$ is
the directional derivative of f at p in
the direction v .

- $\mathcal{X}(M)$ is a $C^\infty(M)$ -module. Though
 $[X, Y]$ is $\mathbb{R}$ -bi-linear in both variables,
it is not $C^\infty(M)$ -bi-linear. We have

$$[fX, gY] = fg[X, Y] + (f \cdot X(g))Y - (g \cdot Y(f))X.$$

In particular $\mathcal{L}_{fX} Y \neq f \mathcal{L}_X Y$, where

$$\mathcal{L}_X(Y) := [X, Y] \text{ is defined to be}$$

the Lie derivative of Y with respect to X .

- For $X = \sum_i f_i \frac{\partial}{\partial x_i}$, $Y = \sum_j g_j \frac{\partial}{\partial x_j}$, on (U, φ) ,

$$[X, Y] = \sum_j \left(\sum_i \left[f_i \frac{\partial g_j}{\partial x_i} - g_j \frac{\partial f_i}{\partial x_i} \right] \right) \frac{\partial}{\partial x_j}.$$

11-6
Exercise 1. Let A be a (commutative) ring, $\delta_1, \delta_2 \in \text{Der}(A)$. Then $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \text{Der}(A)$.

2. $[\delta_1, \delta_2] = -[\delta_2, \delta_1]$; for $\delta_3 \in \text{Der}(A)$,

$$[\delta_1, [\delta_2, \delta_3]] + [\delta_2, [\delta_3, \delta_1]] + [\delta_3, [\delta_1, \delta_2]] = 0.$$

The identity (2) is called the Jacobi identity.

• For $x, y, z \in \mathcal{X}(M)$ it thus follows

$$[x, y] = -[y, x] \text{ \& Jacobi identity holds.}$$

Defⁿ: A real Lie algebra is an $\mathbb{R}$ -vector space L , with an $\mathbb{R}$ -bilinear map $[\cdot, \cdot]: L \times L \rightarrow L$ satisfying (i) $[x, y] = -[y, x]$ $\forall x, y \in L$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in L$.

Examples. 1. Any vector space L over $\mathbb{R}$ with $[\cdot, \cdot] \equiv 0$ is a Lie algebra, called an abelian Lie algebra.

2. A be any associative $\mathbb{R}$ -algebra (e.g. $A = M_n(\mathbb{R})$). Define $[a, b] = ab - ba$. Then $(A, [\cdot, \cdot])$ is a Lie algebra.

3. $sl(n, \mathbb{R})$, $O(n, \mathbb{R})$, $su(n)$, $u(n)$ are Lie algebras,

¹¹⁻⁷ here we use the same Lie bracket as (2).

Remark :- Let M, N be smooth manifolds, $\varphi: M \rightarrow N$ be smooth, let $X \in \mathcal{X}(M)$.

Define, for $p \in M$, $q = \varphi(p) \in N$,

$Y(q) = D\varphi(p)(X(p))$. One can ask: if φ is a homeomorphism, does this give a smooth vector field on N ?

Example :- Let $M = N = \mathbb{R}$ with x as the canonical coordinate function. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be the map $\varphi(t) = t^3$. Then φ is smooth & a homeomorphism. Define

$$Y(u) = X(t) = D\varphi\left(\frac{d}{dx}\Big|_t\right), \quad u = t^3$$
$$= \frac{d\varphi}{dx}\Big|_t \cdot \frac{d}{dx}\Big|_{\varphi(t)} = 3t^2 \frac{d}{dx}\Big|_u$$

So $Y = 3x^{2/3} \frac{d}{dx}$. The function $x \mapsto x^{2/3}$ is not diff at 0, hence Y is not smooth.

Def :- Let M, N be smooth and $\varphi: M \rightarrow N$ smooth. We say $X \in \mathcal{X}(M)$ & $Y \in \mathcal{X}(N)$ are φ -related if $D\varphi(p)(X(p)) = Y(\varphi(p)) \forall p \in M$.

so

$$D\varphi \circ X = Y \circ \varphi; \text{ i.e. } D\varphi(p)(X(p)) = Y(\varphi(p)) \forall p.$$

11-8
Exercise: If $\varphi: M \rightarrow N$ is a diffeomorphism, then
 for $X \in \mathcal{X}(M)$, $p \in M$, $q = \varphi(p)$, the vector
 field $Y: q \mapsto D\varphi(p)(X(p))$ is smooth.

Lemma: $\varphi: M \rightarrow N$ smooth, $X \in \mathcal{X}(M)$, Y
 $\in \mathcal{X}(N)$ then X, Y are φ related $\Leftrightarrow$
 $\forall g \in C^\infty(N)$, $X(g \circ \varphi) = Y(g) \circ \varphi$.

Pf: X, Y are φ -related $\Leftrightarrow D\varphi \circ X = Y \circ \varphi$
 We have, for $p \in M$ & $g \in C^\infty(N)$,

$$D\varphi(p)(X(p))(g) = Y(\varphi(p))(g)$$

$$\Leftrightarrow X(p)(g \circ \varphi) = Y(g)(\varphi(p))$$

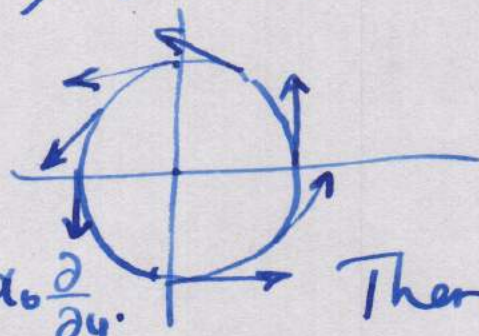
$$\Leftrightarrow X(g \circ \varphi)(p) = (Y(g) \circ \varphi)(p)$$

$$\Leftrightarrow X(g \circ \varphi) = Y(g) \circ \varphi. \quad \square$$

Example from the book: Let $\varphi: \mathbb{R} \rightarrow S^1 \subseteq \mathbb{R}^2$
 be $\varphi(t) = (\cos t, \sin t)$. Let $X \in \mathcal{X}(\mathbb{R})$ be
 $X = \frac{d}{dt}$, t coord on $\mathbb{R}$; $Y \in \mathcal{X}(\mathbb{R}^2)$ be

$$Y = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

i.e. $Y((x_0, y_0)) = -y_0 \frac{\partial}{\partial x} + x_0 \frac{\partial}{\partial y}$. Then $Y|_{S^1}$
 is a vector field on S^1 , φ related to X
Exercise: Write details.



11-3
Vector fields on Lie groups :- In the last example

We had $M = S^1$, a Lie group and the vector field Y on it. Note that $Y(p)$ was obtained from $Y(e)$, $e = (1, 0) = \text{identity elem of } S^1$, by 'rotation' corresponding to $p \in S^1$. To be precise, recall that in a Lie group G $\mu: G \times G \rightarrow G$ & $i: G \rightarrow G$ resp. the mult. & inversion, are smooth. It follows that for $g \in G$, the maps $L_g: G \rightarrow G$, $x \mapsto gx$ and $R_g: x \mapsto xg$, are both smooth. Let $v \in T_e G$ and $g \in G$. Define X by

$$X(g) = DL_g(v) \equiv DL_g(e)(v) \in T_g(G).$$

- Let (U, α) be a chart $\ni e$ and (V, β) a chart $\ni g$. Write $v = \sum_j a_j \frac{\partial}{\partial x_j}$, $a_j \in \mathbb{R}$.

$$\begin{aligned} \text{Then } X(g) &= DL_g^{(v)}(v) = DL_g^{(e)}\left(\sum_j a_j \frac{\partial}{\partial x_j}\right) \\ &= \sum_j a_j DL_g^{(e)} \frac{\partial}{\partial x_j} = \sum_j a_j \sum_i DL_g^{(e)}(y_i) \frac{\partial}{\partial y_i} \\ &= \sum_j a_j \sum_i DL_g^{(e)}\left(\frac{\partial}{\partial x_j}\right)(y_i) \frac{\partial}{\partial y_i} \end{aligned}$$

11-10

$$= \sum_{j,i} a_j \frac{\partial}{\partial x_j} (y_i \circ L_g) \frac{\partial}{\partial y_i} \quad \text{on some open subset of } V \text{ containing } g.$$

Now L_g is a diffeomorphism, hence $y_i \circ L_g$ is C^∞ and, for each j , $a_j \frac{\partial}{\partial x_j} (y_i \circ L_g)$ is C^∞ . Hence X is C^∞ , i.e. $X \in \mathcal{X}(G)$.

→ Intuitively we feel that X , which we constructed by translating around v on G , should be 'translation invariant'.

• Def: We call a vector field $X \in \mathcal{X}(G)$ left invariant if X is L_g related to itself $\forall g \in G$, i.e. $DL_g \cdot X = X \circ L_g \quad \forall g \in G$.

• If X is left invariant, then $DL_g(X(e)) = X \circ L_g(e) = X(g)$.

• If Y is another left inv. vector field $\in \mathcal{X}(G)$ with $Y(e) = X(e)$, then $Y = X$. Since

$$X(g) = DL_g(X(e)) = DL_g Y(e) = Y(g).$$

• $\mathcal{L} =$ Set of all left inv. vector fields on G .
 is an $\mathbb{R}$ -vector subspace of $\mathcal{X}(G)$.

$\mathcal{L} \rightarrow T_e(G)$, $X \mapsto X(e)$ is an $\mathbb{R}$ -isom.

Hence $\dim \mathcal{L} = \dim G = \dim T_e G$. —x—x—

Lecture-12

Summary of work so far:- Let us quickly summarize what we have discussed so far in the course.

- Differentiable manifolds: To study geometric objects like curves, surfaces intrinsically, not as embedded objects in some Euclidean space. Locally the discussion can be brought down to the Euclidean setting using the Coordinate chart maps.
- Smooth functions: between smooth manifolds are defined using the notion in Euclidean spaces, using Chart maps, transition functions facilitate this.
- Need of calculus on manifolds: Geometry is bothered by studying shapes of objects. One way to do that is to see how the object 'turns' or 'curves' around each of its points. From the Euclidean experience, this is done by measuring the 'rate of change of tangent spaces' on the object (smooth!), which naturally requires the notion of derivative of a smooth map; To talk of 'curvature', metric etc.
- Tangent vectors / Tangent spaces: We need 'direction' to talk of derivative of a function. Since we do not have an ambient Euclidean space, we achieve this by borrowing vectors from $\mathbb{R}^n$ locally:

- ¹²⁻² • We used curves in the Euclidean case to define tangent vectors; if $U \subseteq \mathbb{R}^n$ open, $p \in U$, $\gamma: (-\epsilon, \epsilon) \rightarrow U$ a curve in U , $\gamma(0) = p$. Then $\dot{\gamma}(0) = (\dot{\gamma}_1(0), \dots, \dot{\gamma}_n(0))$ is defined as a tangent vector to U at p .

This inspires the defⁿ: $T_p(U) =$ Set of equiv. classes of curves in U , under the equiv. relⁿ $\sigma \sim \tau$ if $\sigma(0) = \tau(0) = p$, $\dot{\sigma}(0) = \dot{\tau}(0)$; $[\sigma] :=$ Class of σ .

Tangent vectors as differential operators: For $U \subseteq \mathbb{R}^n$ as above, we prove $T_p(U) = \mathbb{R}^n$, since any $v \in \mathbb{R}^n$ (assuming $p = \text{origin}$) can be realized as $\dot{\gamma}(0)$, e.g. (γ a straight line) $\cap U$.

Given $v \in \mathbb{R}^n$ and $f \in C^\infty(p)$, we think of v as a diff. operator by setting $v(f) :=$ directional derivative of f at p , in the direction of v .

This gives a natural $T_p(U) \cong \mathbb{R}^n \cong \text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$.

- The directional derivative $D_v f(p)$ can be computed as $\frac{d}{dt} (f \circ \gamma(t)) \Big|_{t=0}$ for any curve γ on U with

$\gamma(0) = p$, $\dot{\gamma}(0) = v$. Hence the operator v on $C^\infty(p)$ can be identified with the equiv. class $[\gamma]$ as above; i.e.

$$T_p(U) \cong \text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R}) \cong \{[\gamma] \mid \gamma(0) = p\} \xrightarrow{\sim}$$

¹²⁻³ • The first identification is because we proved (Lecture 8) that any $\delta \in \text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$, $\exists! v \in \mathbb{R}^n \equiv T_p(u)$, s.t. $\delta = [v]$.

• Under these identifications, the (standard) basis vectors of $\mathbb{R}^n$, e_i correspond to the directional derivative operator at $(p=0)$ in the direction e_i , i.e. $\frac{\partial}{\partial x_i}$. So $\left\{ \frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p \right\}$ form a basis of $T_p(u)$.

• We define for M smooth, $p \in M$, a tangent vector v at p , as an element of $\text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$, i.e. $T_p M := \text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$.

• If $(U, \varphi) \ni p$ is a coordinate chart, $\varphi(p) = 0$ and $\{u_i\}_{1 \leq i \leq n}$ the canonical coordinates in $\mathbb{R}^n$,

$x_i := u_i \circ \varphi$, we define for $f \in C^\infty(p)$,

$$\frac{\partial}{\partial x_i}|_p(f) := \frac{\partial}{\partial u_i}(f \circ \varphi^{-1})(0) \equiv \frac{\partial}{\partial u_i}(f \circ \varphi').$$

$\left\{ \frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p \right\}$ form a basis of $T_p(M)$;

for $v \in \text{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R}) = T_p(M)$, we have

$$v = \sum_i v(x_i) \frac{\partial}{\partial x_i}|_p$$

• For $\gamma: (-\epsilon, \epsilon) \rightarrow M$ smooth, $\gamma(0) = p$, define

12-4. $\dot{\gamma}(0) \in T_p M$ by $\dot{\gamma}(0)(f) := \frac{d}{dt}(f \circ \gamma)(0) = \frac{d}{dt} f(\gamma(t)) \Big|_{t=0}$,
 here $f \in C^\infty(p)$ is arbitrary.

• Derivatives of smooth maps: M, N smooth,
 $f: M \rightarrow N$ smooth, $p \in M$. Then we would like
 to generalize the notion of derivative $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$.
 Which, for $p \in \mathbb{R}^m$, is a linear map: $T_p \mathbb{R}^m \rightarrow T_{f(p)} \mathbb{R}^n$.

We define $Df(p): T_p M = \text{Der}_{\mathbb{R}}(C_M^\infty(p), \mathbb{R}) \rightarrow T_{f(p)} N = \text{Der}_{\mathbb{R}}(C_N^\infty(f(p)), \mathbb{R})$

by ~~$AF(f)$~~ $Df(p)(v)(g) = v(g \circ f)$ for
 $g \in C_N^\infty(f(p))$ arbitrary. Equivalently if

$v \in T_p M$, $\exists$ a smooth curve γ on M , $\gamma(0) = p$

& $v = \dot{\gamma}(0)$ (as an elem. of $\text{Der}_{\mathbb{R}}(C_M^\infty(p), \mathbb{R})$).

Then $f \circ \gamma \subseteq N$ is smooth, $f \circ \gamma(0) = f(p)$.

Define $Df(p)(v) = (f \circ \gamma)'(0)$. Then

$Df(p): T_p M \rightarrow T_{f(p)} N$ is a linear map, gen.
 the Euclidean notion.

• Jacobian matrix :- $Df(p) \equiv \left(\frac{\partial}{\partial x_i} \Big|_p (y_j \circ f) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$

$(V, \psi) \ni q = f(p)$ a coord. chart in N with
 $\{y_1, \dots, y_n\}$ as coord. functions on V .

• $D(g \circ f)(p) = Dg(f(p)) \circ Df(p)$ (Chain rule)
 $p \in M \xrightarrow{f} N \xrightarrow{g} W$.

12-5. Special case: $M = (-\epsilon, \epsilon) \subseteq \mathbb{R}$, chart $u: t \mapsto t$.

$T_0 M = \mathbb{R} \cdot \frac{d}{du}|_0$. If N smooth, $f \in C^\infty(N)$, $p \in N$,

$v \in T_p N$. Then $Df(p)(v) \in T_{f(p)} \mathbb{R} = \mathbb{R} \cdot \frac{d}{du}|_{f(p)}$ &

$$Df(p)(v) = v(f) \cdot \frac{d}{du}|_{f(p)}.$$

Cotangents, Cotangent space: M smooth, $p \in M$.

$(T_p M)^* =$ Cotangent space to M at p . For

$f \in C^\infty(M)$, $Df(p): T_p M \rightarrow T_{f(p)} \mathbb{R} = \mathbb{R}$, so

$Df(p) \in (T_p M)^*$ and for $v \in T_p M$,

$Df(p)(v) = v(f)$. When treat f as an element of $(T_p M)^*$, we denote $Df(p)$ by $df(p)$. We write $dx_i(p)$ for x_i coords around p , then $dx_i(p) = \left(\frac{\partial}{\partial x_i} \Big|_p \right)^* \in (T_p M)^*$ make the dual basis.

• $df: M \rightarrow \bigcup_p (T_p M)^*$, $f \in C^\infty(M)$ has a local

expression $df(x) = \frac{\partial f(x)}{\partial x_1} dx_1 + \dots + \frac{\partial f(x)}{\partial x_m} dx_m$

for $x \in U$, $(U, \varphi) \ni p$ a local chart.

Defined Submanifolds, Submersions, immersions.
Submersions & immersions on manifolds are canonical locally.



- 12-6 • Vector fields: Tangent vectors are 'local' diff. operators, acting on the algebra of (germs of) locally smooth functions on M . Vector fields are global diff operators, acting on the alg. of globally smooth functions on M , i.e. elements of $\text{Der}_{\mathbb{R}}(C^{\infty}(M), C^{\infty}(M))$. These are maps $X: M \rightarrow T(M) = \bigcup_p T_p M$, $X(p) \in T_p M$ such that, for any coord. chart $U \ni p$ with coordinates $x_1, \dots, x_m$, we have

$$X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i} \Big|_q, \quad q \in U;$$

f_i are smooth.

- $\mathcal{X}(M)$ = $\mathbb{R}$ -vector space of smooth vect. fields on M , is closed under the $\mathbb{R}$ -bilinear map $[\cdot, \cdot]: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$, $[X, Y]$ is given by $[X, Y] = X \circ Y - Y \circ X$; for $p \in M$, $f \in C^{\infty}(M)$, this yields

$$[X, Y](f)(p) = [X, Y](p)(f) = X(p)(Y(f)) - Y(p)(X(f)).$$

- Lie derivatives: For $X \in \mathcal{X}(M)$, when treated as an $\mathbb{R}$ -derivation: $C^{\infty}(M) \rightarrow C^{\infty}(M)$,

¹²⁻⁷ We write $\mathcal{L}_X(f) := X(f)$, $f \in C^\infty(M)$ and call it the Lie derivative of f with respect to X . When $p \in M$, we have $X(p) \in T_p M$.

Given $f \in C^\infty(M)$, $p \in M$, what is $\mathcal{L}_X(f)(p)$?

Following notations, we see that

$$\mathcal{L}_X(f)(p) = X(p)(f) \leftarrow \text{directional}$$

derivative of f at p , in the direction $X(p)$.

Hence we may think of a vector field X as built from the local directional derivative operators $X(p)$, $p \in M$.

- Lie algebras and vector fields :- A Lie alg/ $\mathbb{R}$ is a vector space L over $\mathbb{R}$, with a $\mathbb{R}$ -bilinear map $[\cdot, \cdot]$, called the Lie bracket, $[\cdot, \cdot]: L \times L \rightarrow L$ satisfying $[x, y] = -[y, x]$; Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, $x, y, z \in L$.
• $(\mathcal{X}(M), [\cdot, \cdot])$ is a Lie algebra.

Exercise :- 1. L be a Lie alg, $[\cdot, \cdot]$ its Lie bracket.

Let $\alpha \in L$ be fixed. Define $\text{ad}(\alpha): L \rightarrow L$

by $\text{ad}(\alpha)(y) := [\alpha, y]$. Prove that $\text{ad}(\alpha)$ is a derivation on L , i.e. satisfies the

12-8 Product rule $\text{ad}(x)([y, z]) = [\text{ad}(x)(y), z] + [y, \text{ad}(x)(z)]$.

→ Jacobi identity is equivalent to the above.

2. L be a Lie algebra. We then have the $\mathbb{R}$ -linear map $\text{ad}(x): L \rightarrow L$ for any $x \in L$.

Thus an $\mathbb{R}$ -linear map $\text{ad}: L \rightarrow \text{End}_{\mathbb{R}}(L)$

$x \mapsto \text{ad}(x)$. Prove that 'ad' is a Lie alg. homomorphism, i.e. is additive, ~~$\mathbb{R}$ -linear~~

$$\text{ad}([x, y]) = \text{ad}(x) \circ \text{ad}(y) - \text{ad}(y) \circ \text{ad}(x)$$

$\forall x, y \in L$.

$$[\text{ad}(x), \text{ad}(y)] \text{ in } \text{End}_{\mathbb{R}}(L).$$

• Vector fields on Lie groups :- On a Lie group

G , the translation maps $L_g: G \rightarrow G$ are diffeomorphisms. Given $v \in T_e G$, $g \in G$

we have $DL_g(e): T_e G \rightarrow T_g G$, so

$DL_g(e)(v) \in T_g G$. We define a vector

field X on G by $X(g) := DL_g(e)(v)$.

Then X is left-invariant; ~~see~~.

Call $Y \in \mathfrak{X}(G)$ left-invariant if

$$DL_g \circ Y = Y \circ L_g.$$

→

12-3.

- Any left invariant vector field on G is completely determined by its value at e .
- $\mathcal{G} := \mathbb{R}$ -vector space of all left-inv. smooth vector fields on G .

The map $\mathcal{G} \rightarrow T_e G, X \mapsto X(e)$ is an isomorphism of vector spaces; hence $\dim \mathcal{G} = \dim T_e G = \dim G$. $\square$

Remark: What we did with left translⁿ maps, can be done, similarly, with right translⁿ on G .

Recall: Given $\varphi: M \rightarrow N$ smooth, $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$, we say Y is φ -related to X if $D\varphi \circ X = Y \circ \varphi$, i.e.

$$D\varphi(p)(X(p)) = Y(\varphi(p)) \quad \forall p \in M.$$

We need

Lemma: $X_1, X_2 \in \mathcal{X}(M)$ be φ -related to $Y_1, Y_2 \in \mathcal{X}(N)$ respectively. Then $[X_1, X_2]$ is φ -related to $[Y_1, Y_2]$.

¹²⁻¹⁰
Pf: We proved $X \in \mathfrak{X}(M)$ is φ -rel to $Y \in \mathfrak{X}(N)$
 $\Leftrightarrow X(g \circ \varphi) = Y(g) \circ \varphi \quad \forall g \in C^\infty(N)$. We compute

$$\begin{aligned} X_1 X_2 (g \circ \varphi) &= X_1 (X_2 (g \circ \varphi)) = X_1 (Y_2(g) \circ \varphi) \\ &= Y_1 (Y_2(g)) \circ \varphi \\ &= Y_1 Y_2 (g) \circ \varphi \end{aligned}$$

and similarly $X_2 X_1 (g \circ \varphi) = Y_2 Y_1 (g \circ \varphi)$.

Thus $[X_1, X_2](g \circ \varphi) = [Y_1, Y_2](g \circ \varphi), \forall g$.

$\therefore [X_1, X_2]$ is φ -related to $[Y_1, Y_2]$. ~~□~~

Lie algebra of a Lie group: We use the lemma to $\mathfrak{g} = \{\text{left inv. vector fields on } G\}$.

$$= \{X \in \mathfrak{X}(G) \mid X \text{ is } L_g\text{-related to itself } \forall g \in G\}.$$

Hence it follows that $\forall X, Y \in \mathfrak{g}$,
 $[X, Y] \in \mathfrak{g}$, i.e. $\mathfrak{g}$ is a Lie-subalgebra of $\mathfrak{X}(G)$, called the Lie algebra of G , also denoted by $\text{Lie}(G)$. This plays a big role in the structure theory of Lie groups & their classification.

————— x ————— x ————— x ————— x —————

Lecture-13

Lie algebra of a Lie group :- We ended Lecture-11 by proving: if G is a Lie group and $\mathfrak{g}$ its Lie algebra, i.e. $\mathfrak{g} = \{X \in \mathcal{X}(G) \mid X \text{ is left inv.}\}$, then the map $\mathfrak{g} \rightarrow T_e G, X \mapsto X(e)$ is an isomorphism of vector spaces.

→ If $X, Y \in \mathfrak{g}$, then $[X, Y] \in \mathfrak{g}$ and $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra. We transfer this Lie algebra structure to $T_e G$ by defining for $X, Y \in \mathfrak{g}$; $[X(e), Y(e)] := [X, Y](e)$.

→ This gives $T_e G$ a Lie algebra structure and we sometimes identify $\mathfrak{g}$ with $T_e G$ via the above; $\mathfrak{g} \rightarrow T_e G, X \mapsto X(e)$ is a Lie algebra isomorphism.

Example :- Consider the Lie group $G = GL_n(\mathbb{R})$. Since $G \subseteq \mathbb{R}^{n^2} \simeq M_n(\mathbb{R})$ is an open set, namely $\{X \mid \det X \neq 0\}$, we have $\dim G = n^2$ and $T_I G$ can be identified with $M_n(\mathbb{R})$.

The (global) coordinates on $\mathbb{R}^{n^2}$ are to be labelled as $x_{11}, x_{12}, \dots, x_{n1}, \dots, x_{nn}$. →

¹³⁻² Then, on GL_n , $\left\{ \frac{\partial}{\partial x_{11}}|_I, \dots, \frac{\partial}{\partial x_{ij}}, \dots, \frac{\partial}{\partial x_{nn}}|_I \right\}$
 form a basis of $T_I(GL_n)$. We know that
 any $v \in T_I G$ has! expression $v = \sum_{ij} v_{ij} \frac{\partial}{\partial x_{ij}}|_I$
 and $v_{ij} = v(x_{ij})$. By the identification of
 $T_I G$ with $\mathbb{R}^{n^2}$ & in turn with $M_n(\mathbb{R})$, we
 get $T_I GL_n \rightarrow M_n(\mathbb{R})$.
 $\text{Lie}(GL_n) \cong T_I GL_n \xrightarrow{v \mapsto (v(x_{ij}))_{ij}}$

We have an issue here to settle! We
 have a Lie algebra structure on $T_I GL_n$,
 a Lie alg. structure on $M_n(\mathbb{R})$. The above
 is, as of now, only a vector space map.
 We'll prove: the above is a Lie alg. isom.

Proof: Let $X, Y \in \text{Lie}(GL_n)$; so these are
 left-inv. vector fields on GL_n . These give
 $X(e), Y(e) \in M_n(\mathbb{R})$ by the above recipe,
 also we have $[X, Y](e) \in M_n(\mathbb{R})$, where
 $[X, Y]$ is the Lie bracket on $\mathcal{X}(GL_n)$. To
 show the above map is a Lie alg map, it
 suffices to check: $[X, Y](e)_{ij} = [X(e), Y(e)]_{ij}$.

B-3. Now $[X, Y](e)_{ij} = [X, Y](e)(x_{ij})$
 $= X(e)Y(x_{ij}) - Y(e)X(x_{ij}).$

We have $X = \sum X(x_{ij}) \frac{\partial}{\partial x_{ij}}, Y = \sum Y(x_{ij}) \frac{\partial}{\partial x_{ij}}.$

Also, for $a \in GL_n$,

$X(x_{ij})(a) = X(e)(x_{ij} \circ L_a)$ by left inv.

$\rightarrow \begin{cases} X(x_{ij})(a) = DL_a(X(e))(x_{ij}) = X(e)(x_{ij} \circ L_a), \\ Y(x_{ij})(a) = Y(e)(x_{ij} \circ L_a). \end{cases}$ We need to

Compute also $x_{ij} \circ L_a(g)$, for $g \in GL_n$

$x_{ij}(ag) = \sum_k x_{ik}(a) x_{kj}(g).$ Therefore

$\rightarrow x_{ij} \circ L_a = \sum_k x_{ik}(a) x_{kj}.$ Hence

$Y(x_{ij})(a) = Y(e)(x_{ij} \circ L_a) = Y(e)\left(\sum_k x_{ik}(a) x_{kj}\right)$
 $= \sum_k x_{ik}(a) Y(e)(x_{kj}).$

Hence $Y(x_{ij}) = \sum_k x_{ik} \cdot Y(e)(x_{kj})$ as functions.

Finally, $[X, Y](e)_{ij} = X(e)\left(\sum_k x_{ik} Y(e)(x_{kj})\right)$
 $- Y(e)\left(\sum_k x_{ik} X(e)(x_{kj})\right) = \sum_k X(e)(x_{ik}) Y(e)(x_{kj})$
 $- \sum_k Y(e)(x_{ik}) X(e)(x_{kj}) = [X(e), Y(e)]_{ij}. \quad \square$

Exercise: Check that $\mathfrak{sl}_n, \mathfrak{u}(n), \mathfrak{su}(n), \mathfrak{o}(n), \mathfrak{so}(n)$ are all Lie subalgebras of the corresponding matrix algebra, i.e. each of the above is closed under the matrix Lie bracket. Recall the following: Lie group / Lie algebra / Condition

$GL_n(\mathbb{R})$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$	—
$SL_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{R})$	$\text{Trace}(X) = 0$
$O(n, \mathbb{R})$	$\mathfrak{o}(n)$	$X^t = -X$
$SO(n, \mathbb{R})$	$\mathfrak{so}(n)$	" — "
$U(n)$	$\mathfrak{u}(n)$	$X^* = -X$
$SU(n)$	$\mathfrak{su}(n)$	$X^* = -X \text{ \& } \text{tr}(X) = 0$
$Sp(2n, \mathbb{R})$	$\mathfrak{sp}(2n)$	$JX^t J = X,$ $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$

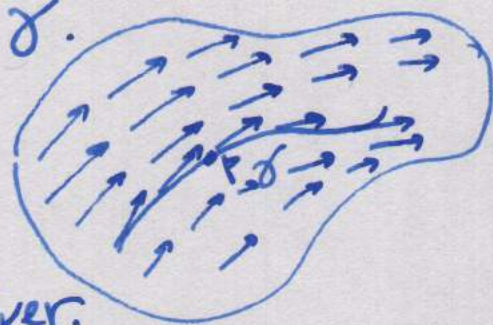
Integral curves and flows of vector fields:

Let M be a smooth manifold, $p \in M$. Let $v \in T_p(M)$. Then using local charts, we have shown that $\exists$ a smooth curve γ passing through p and $\dot{\gamma}(0) = v$. More precisely, $\gamma: (-\epsilon, \epsilon) \rightarrow M$, smooth, $\gamma(0) = p$ & $\dot{\gamma}(0) = v$ (as a derivation on $C^\infty(p)$). Let X be a smooth vector field on M . Then, the above gives us γ s.t. $\gamma(0) = p$ & $\dot{\gamma}(0) = X(p)$.

13-5. A priori, γ works for $p = \gamma(0)$, but if $q \in M$ is on γ , say $\gamma(t_1) = q$, we may not have $\dot{\gamma}(t_1) = X(q)$.

Defⁿ For $X \in \mathcal{X}(M)$ and $p \in M$, a smooth curve γ is called an integral curve of X , through p if $\gamma(0) = p$ and $\dot{\gamma}(t) = X(\gamma(t))$ for all t in the domain of γ .

→ You may think of γ as the path traced by a blade of grass on the surface of a flowing river.



Thus, the curve γ as it fits the defⁿ fits the vector field for an interval of time.

Integral curves are instrumental in studying vector fields on M , comparing tangent spaces at different points, in differential calculus on M .

Given $X \in \mathcal{X}(M)$ & $p \in M$, does X admit an integral curve through p ?

Suppose $(U, \varphi) \ni p$ is a coordinate chart on M , γ an integral curve of X through p . Then, by defⁿ,

$$X(\gamma(t)) = \dot{\gamma}(t) = \sum_{i=1}^m X(x_i) \frac{\partial}{\partial x_i} \Big|_{\gamma(t)}.$$

$$\text{and } \dot{\gamma}(t_0) = \sum_{i=1}^m \frac{d}{dt} (x_i \circ \gamma) \Big|_{t=t_0} \cdot \frac{\partial}{\partial x_i} \Big|_{\gamma(t_0)} \quad \rightarrow$$

¹³⁻⁶ Hence the question whether X admits an integral curve through p reduces to solving the system of ODE

$$\frac{d}{dt} (x_i \circ \gamma) = X(x_i), \quad t \in (-\epsilon, \epsilon); \quad 1 \leq i \leq m$$

Using local coordinates around $p \in (U, \varphi)$, this reduces to finding $\alpha: (-\epsilon, \epsilon) \rightarrow V = \varphi(U)$

satisfying $\frac{d}{dt} \alpha = X(\alpha)$, with initial value $\alpha(0) = 0$ (*) (we assume $\varphi(p) = 0$);

We now apply:

Thm (Prop. 1.9.1, Kum) (Existence of sol's of ODE):

Let $V \subseteq \mathbb{R}^m$ be open, $X: V \rightarrow \mathbb{R}^m$ be Lipschitz;

$\|X(x) - X(y)\| \leq M \|x - y\| \quad \forall x, y \in V$. Let $x_0 \in V$, $B(x_0, r) \subset V$, $C > 0$ be such that $\|X(x)\| \leq C \quad \forall x \in B(x_0, r)$. Let $\epsilon < \min \{ \frac{1}{M}, \frac{r}{C} \}$. Then $\exists!$ C^1 curve $\alpha: [-\epsilon, \epsilon] \rightarrow B(x_0, r)$

satisfying $\dot{\alpha}(t) = X(\alpha(t))$, $\alpha(0) = x_0$. $\square$

to get a unique (local) solution to (*).

Example: Let $M = \mathbb{R}^2$, $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ and

$p = (a, b)$. If we write $\gamma(t) = (x(t), y(t))$. Then we must solve $\dot{x} = -y$, $\dot{y} = x$, $x(0) = a$, $y(0) = b$

13-7 To find the integral of X through p , we

solve $\frac{dx}{dt} = -y, \frac{dy}{dt} = x$, with initial

values $x(0)=a$ & $y(0)=b$ resp. This has

solⁿ $x(t) = a \cos t - b \sin t, y(t) = a \sin t + b \cos t$.

When $(a,b) \neq (0,0)$, we get a circle of radius $\sqrt{a^2+b^2}$. (do the details). (See Kum. for more).

Flow of a vector field :- We can rephrase the theorem on existence & uniqueness for solⁿ of a system of ODE's to get better results:

Thm (Local flows): Let $V' \subseteq \mathbb{R}^m$ be open, $p \in V'$.

Let $X: V' \rightarrow \mathbb{R}^m$ be $C^k, k \geq 1$. Then $\exists \epsilon > 0$ & open subset $V \subseteq V', p \in V$, and a unique

C^k -map $F: (-\epsilon, \epsilon) \times V \rightarrow V'$ such that

(i) $\frac{d}{dt}(F(t,x)) = X(F(t,x))$ (ii) $F(0,x) = x$

$\forall t \in (-\epsilon, \epsilon) \text{ \& } x \in V$. (see Th. 2.8.9. Kum). $\square$

— This gives, for a smooth manifold

Thm (Existence of a local flow) :- Let X be a C^{k-1} vector field on a C^k -manifold M ($k \geq 2$)

and $p \in M$. There exists an open interval

$I \ni 0$, open set $p \in U \subseteq M$ such that $\exists!$ local flow $F: I \times U \rightarrow M$ for X at p , $F \in C^{k-1}$.

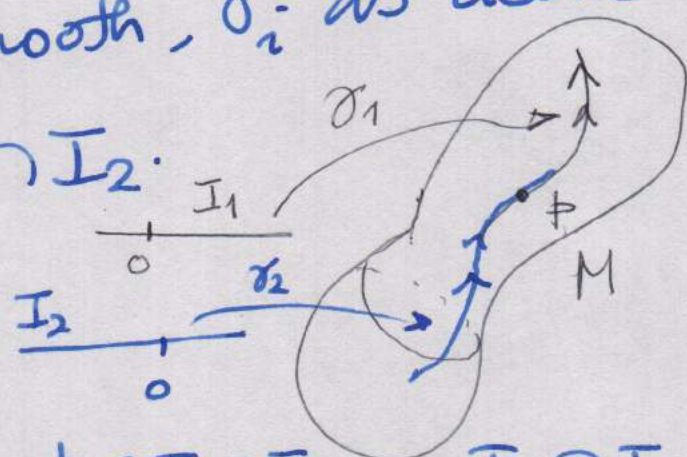
13-8. i.e. if $F_1: I \times U \rightarrow M$ & $F_2: I \times U \rightarrow M$ are both local flows with domain $I \times U$, at t , then $F_1 = F_2$. $\square$

F is called the flow of the vector field X . Hence a local flow for X is a family of integral curves for all points in some open set U around p , such that all these curves have the same domain I , independent of the initial condition $p \in U$.

We would like to compare two integral curves of X , $\gamma_1: I_1 \rightarrow M$, $\gamma_2: I_2 \rightarrow M$, $\gamma_i(0) = p$, $i=1,2$, $\dot{\gamma}_i(0) = X(p)$. We have

Lemma:- X, M be smooth, γ_i as above.

Then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.



Proof:- Consider $I_1 \cap I_2$.

Since both are intervals, $p \in I_1 \cap I_2 \Rightarrow I_1 \cap I_2$ is an interval. The set $\{t \in I_1 \cap I_2 \mid \gamma_1(t) = \gamma_2(t)\}$ is both open and closed in $I_1 \cap I_2$. etc. $\square$

Let $\{\gamma_j\}_{j \in J}$ be the family of integral curves of X with $\gamma_j: I_j \rightarrow M$, $\gamma_j(0) = p \neq j$. Then

¹³⁻⁹ The Lemma allows us a unique $\gamma: \bigcup_{j \in J} I_j \rightarrow M$ extending γ_j on $I_j \forall j$. Then γ is a smooth curve through p , is an integral curve of X , with largest (maximal) domain $I_p := \bigcup_j I_j$, we call γ the maximal integral curve of X through p , denote it by γ_p .

Example! For $M = \mathbb{R}^2$ and $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and $p = (x_0, y_0)$, γ_p is the circle

$\gamma_p(t) = (x(t), y(t))$, with

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \end{pmatrix}$$

and $I_p = \mathbb{R}$, for every $p \in \mathbb{R}^2$.

Example! For $M = \mathbb{R}$, X be the vector field $X(x) = (1+x^2) \frac{\partial}{\partial x}$.

Exercise! Check that the maximal integral curve with initial condition $p=0$ is the curve $\gamma: (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ given by

$$\gamma(t) = \tan t$$



We can ask the following natural question:

Given $p_0 \in M$, $\gamma_{p_0}: I_{p_0} \rightarrow M$ be the maximal integral curve for X with initial condition p_0 . For $s \in I_{p_0}$ we have $p_1 = \gamma_{p_0}(s) \in M$.

Hence there is the maximal integral curve

$\gamma_{p_1}: I_{p_1} \rightarrow M$ with initial condition p_1 .

How are γ_{p_0} & γ_{p_1} related? We have

Lemma :- We have $I_{p_1} = I_{p_0} - s$ and

$$\gamma_{p_1}(t) = \gamma_{p_0}(t+s) \quad \forall t \in I_{p_0} - s.$$

In other words, the 'traces' $\gamma_{p_0}(I_{p_0})$ and $\gamma_{p_1}(I_{p_1})$ in M of the max integral curves γ_{p_0} and γ_{p_1} are the same; they differ only by a simple reparameterization $u = t + s$.

PS! Let $\sigma(u) = \gamma_{p_0}(u-s)$. Then $\dot{\sigma}(0) = \dot{\gamma}_{p_0}(-s) = X(\gamma_{p_0}(-s))$. Hence σ is an integral curve of X through $\sigma(0) = \gamma_{p_0}(-s)$ etc.

Def! We call X complete if all max integral curves of X have domain $\mathbb{R}$, incomplete otherwise. $\boxtimes$

Integral curves contd.: One clarification on using the ODE theorem: if $f: V \rightarrow \mathbb{R}^m$ is continuously differentiable, $V \subseteq \mathbb{R}^d$ open. Then for every $p \in V$, $\exists r > 0$, such that all $\frac{\partial f_i}{\partial x_j}$ are bounded by some $M' > 0$. Then for $M := \sqrt{d}m \cdot M'$, one has, $\forall x, y \in B(0, r)$, $\|f(x) - f(y)\| \leq M \cdot \|x - y\|$. $\square$

We defined the notion of the maximal integral curve for a vector field and the notion of a complete vector field, i.e. that whose all max integral curves have domain $\mathbb{R}$, incomplete q.v. Looking ahead; if X is a complete vector field on M , $\{\gamma_p\}$ its max integral curves, we'll see that $\mathbb{R}$ there is a diffeomorphic action of $\mathbb{R}$ on M , i.e. a group homom. $\Phi_X: \mathbb{R} \rightarrow \text{Diff}(M)$, induced from $\mathbb{R} \times M \rightarrow M$ given by $(t, p) \mapsto \gamma_p(t)$. In particular if we write $\varphi_t(p) := \gamma_p(t)$, then

$$\varphi_s \circ \varphi_t = \varphi_{s+t}.$$



¹⁴⁻² This is the analogue of translations on M , and we'll use this to compare tangent spaces at different points on M . $\square$

Local flows :- Let M be smooth and X a smooth vector field on M , $p \in M$. Then $\exists \epsilon > 0$, open $U \subseteq M$, $p \in U$ and a unique smooth map $F: (-\epsilon, \epsilon) \times U \rightarrow M$ with $F(0, x) = x \forall x \in U$, $\frac{d}{dt} (F(x, t)) = X(F(x, t)) \forall t \in (-\epsilon, \epsilon), x \in U$.

In other words $\exists$ a unique family of integral curves of X on U , having the same domain $(-\epsilon, \epsilon)$, passing through pts of U . $\square$

$I \neq$ With the above notation, denote for $(t, z) \in (-\epsilon, \epsilon) \times U$, $\varphi_t(z) := F(t, z)$. If γ_p be as before, $\gamma_p(t) = F(t, p)$ for $t \in I$.

$\rightarrow$ When X is complete, $\varphi_t: M \rightarrow M$ is a diffeomorphism, since $\varphi_s \circ \varphi_t(p) = \varphi_{s+t}(p)$, $\varphi_0(p) = p \forall p \in M$.

Proof :- For a fixed s , we consider the curves $t \mapsto \varphi_s \circ \varphi_t(p)$ and $t \mapsto \varphi_{s+t}(p)$.

14-3.

Then both have initial condition $= \varphi_s(p)$ and are integral curves of X . Hence they must be equal (having domains $= \mathbb{R}$). Inverse of φ_t is φ_{-t} clearly. $\square$

Defⁿ: $\{\varphi_t\}_{t \in \mathbb{R}}$ is called the one parameter group of diffeomorphisms associated to X .

Even when X is not complete one can define a local notion:

Defⁿ: A local 1-parameter family of local diffeomorphisms is a collection $\{U_\alpha, \varphi_t^\alpha, \epsilon_\alpha\}_\alpha$ satisfying: (i) U_α are open, $\bigcup U_\alpha = M$
(ii) $\epsilon_\alpha > 0 \forall \alpha$ (iii) domain of $\varphi_t^\alpha \supseteq U_\alpha$ and the map $(t, z) \mapsto \varphi_t^\alpha(z)$ for $z \in U_\alpha$ & $t \in (-\epsilon_\alpha, \epsilon_\alpha)$, is smooth.

(iv) $\varphi_t^\alpha(z) = \varphi_t^\beta(z) \forall z$ in $\text{domain}(\varphi_t^\alpha) \cap \text{dom}(\varphi_t^\beta)$
 Δt with $|t| < \min\{\epsilon_\alpha, \epsilon_\beta\}$.

(v) for t, s with $t, s, t+s \in (-\epsilon_\alpha, \epsilon_\alpha)$, $\varphi_t^\alpha \circ \varphi_s^\alpha$ is defined on an open set $\supseteq U_\alpha$ & $\varphi_t^\alpha \circ \varphi_s^\alpha = \varphi_{t+s}^\alpha$ on U_α . $\rightarrow$

14-4.

• Examples / Exercise : 1. Let $M = \mathbb{R}$. Then $\varphi_t: \mathbb{R} \rightarrow \mathbb{R}$ given by (i) $\varphi_t(x) = x + t$ (ii) $\varphi_t(x) = e^t x$ are both one-parameter groups of diffeoms.

2. $M = \mathbb{R}^2$, $\varphi_t(x, y) = (\cos t x - \sin t y, \sin t x + \cos t y)$ is a one-param. group of diffeoms on $\mathbb{R}^2$.

3. $M = \mathbb{R}^n$ and $A \in M_n(\mathbb{R})$ be fixed. Then $t \mapsto \exp(tA) =: \varphi_t$; $x \mapsto \varphi_t(x)$ is a one-param family of diffeoms.

• Lemma : Given a local 1-parameter family $\{\varphi_t^\alpha\}$ of M , $\exists$ a vector field $X \in \mathcal{X}(M)$ such that it generates the family.

Pf:- We define, for $p \in M$ & $X \in \mathcal{X}(M)$,
 $X(p)(f) := \left. \frac{d}{dt} f \circ \varphi_t^\alpha \right|_{t=0}$ Then X indeed is

smooth. (?) . Let $p \in U_\alpha$ and define

$\gamma_p: (-\epsilon_\alpha, \epsilon_\alpha) \rightarrow M$ by $\gamma_p(t) = \varphi_t^\alpha(p)$. Then

γ_p is an integral curve to X , $\gamma_p(0) = p$ etc.

→ For a given X , a local family of diffeoms is not unique nec. ⊠

14-5 Example/exercise:- Let $X \in \mathcal{X}(M)$,

$\psi: M \rightarrow M$ a diffeomorphism. Let Y be the "push forward" vector field, i.e. for $q \in M$,

$q = \psi(p)$, $Y(q) = D\psi(p)(X(p))$. Then $Y \in \mathcal{X}(M)$. If $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}$ is a local

one parameter family of diffeos for X , then $\{(\psi(U_\alpha), \psi \circ \varphi_t^\alpha \circ \psi^{-1}, \epsilon_\alpha)\}_\alpha$ is one for Y .

Complete vector fields: We have some situations which yield complete fields.

- Let $X \in \mathcal{X}(M)$. ~~Let~~ Assume that X admits a local one-param. family $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ of local diffeos with $\inf \epsilon_\alpha > 0$. Then X is complete.

Proof (sketch): Let $\epsilon = \inf \epsilon_\alpha > 0$ and define $\varphi_t(p) := \varphi_t^\alpha(p)$, if $p \in U_\alpha$, $t \in (-\epsilon, \epsilon)$. Extend φ_t to the whole of $\mathbb{R}$ by the rule:

$$\varphi_t := \underbrace{\varphi_{t/n} \circ \dots \circ \varphi_{t/n}}_{n\text{-fold}}, \quad n \text{ is s.t. } t/n \in (-\epsilon, \epsilon).$$

$\leadsto$

14-6. Then φ_t is well defined and $\gamma_p(t) := \varphi_t(p)$ is a max integral curve for X with $\gamma_p(0) = p$ and domain $= \mathbb{R}$. Hence X is complete. $\square$

Vector fields on M compact / Lie groups:

Let M be compact & $X \in \mathcal{X}(M)$.

Lemma: X is complete.

PF: Let $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}$ be a local one-param. family for X . Compactness $\Rightarrow$ finitely many U_α 's cover X , say $U_{\alpha_1}, \dots, U_{\alpha_r}$.

Then $\{(U_{\alpha_i}, \varphi_t^{\alpha_i}, \epsilon_{\alpha_i})\}$ is a 1-param family for X with $\inf \epsilon_{\alpha_i} > 0$. Hence X is complete. $\square$

$\rightarrow$ If M be smooth & $X \in \mathcal{X}(M)$ have compact support, then X is complete.

This follows by working with an open subset containing the support of X , instead of M .



14-7

Lemma :- Let G be a Lie group and $X \in \mathfrak{g}$ i.e. X is a left-inv. vector field on G . Then X is complete.

Proof :- Let $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ be a 1-param. family for X as above. Let $e \in G$ be the identity element of G , $e \in U_{\alpha_0}$. Then the family $\{(L_g(U_{\alpha_0}), L_g \circ \varphi_t^{\alpha_0} \circ L_{g^{-1}}, \epsilon_{\alpha_0}), g \in G\}$ is a local 1-param. family for $DL_g(X) = X$, as X is left-inv. The criterion for completeness is thus satisfied by this family; hence X is complete. $\square$

Exponential map :- Let M be smooth and X a complete smooth vector field on M . Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be the 1-param group of diffeoms associated to X . We define the exponential map of X by $\text{Exp}_x(p) := \varphi_1(p) = \gamma_p(1)$,
so $\text{Exp}_x : M \rightarrow M$. $\leadsto$

Examples :- 1. Let $A = (a_{ij}) \in M_n(\mathbb{R})$. Consider the vector field $\hat{A}$ on $\mathbb{R}^n$ defined by

$$\hat{A} = \sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i}$$

Consider $F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$F(t, x) = e^{tA}(x), \text{ where recall}$$

$$e^{tA} = \sum_j \frac{t^j}{j!} A^j. \text{ Clearly } F \text{ is smooth.}$$

This defines a flow for $\hat{A}$. Setting

$$\varphi_t(x) = e^{tA}x \quad (x \text{ a column vector}),$$

$$\text{we have } \varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}, \quad \varphi_0 = 1.$$

$$\text{Let } \gamma_p(t) = F(t, p) = e^{tA}p. \text{ Then for } f \in C^a(\mathbb{R}^n),$$

$$\begin{aligned} \dot{\gamma}_p(0)(f) &= \frac{d}{dt} \Big|_{t=0} f(e^{tA}p) \\ &= \sum_j \frac{\partial f}{\partial x_j}(Ap)_j \\ &= \sum_{j,k} a_{jk} p_k \frac{\partial f}{\partial x_j}, \text{ where} \end{aligned}$$

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}. \text{ Hence } \dot{\gamma}_p(0) = \hat{A}(p).$$

$$\text{Thus } \text{Exp}_{\hat{A}}(p) = \varphi_1(p) = \gamma_p(1) = e^A \cdot p$$



14-9
2. Lie groups: G be a Lie group, $\mathfrak{g} = \text{Lie}(G)$.

We saw that any $X \in \mathfrak{g}$ is complete, so Exp_X makes sense. We define (note the confusing notation!)

$$\exp(x) := \bar{\text{Exp}}_x(e) = \varphi_1(e),$$

and more generally,

$$\exp(tx) = \varphi_t(e), \text{ where } \{\varphi_t\}$$

is the 1-parameter group assoc. to X .

This defines a map $\mathfrak{g} \xrightarrow{\exp} G$

$$x \mapsto \exp(x),$$

called the exponential map of G .

For $G = \text{GL}_n(\mathbb{R})$, we have seen that $\mathfrak{g} = (M_n(\mathbb{R}), [\cdot, \cdot])$, where $A \in M_n(\mathbb{R})$ corresponds to the vector field $\tilde{A} = \sum_{i,j} a_{ij} \frac{\partial}{\partial a_{ij}}$.

We then have the 1-parameter group $\{\varphi_t\}$ associated to $\tilde{A}$, and $\text{Exp}_{\tilde{A}}(e) = \varphi_1(e)$

$\exp(x) = \exp(\tilde{A})$ is the exponential of $\tilde{A}$. While

$$\exp(t\tilde{A}) = \varphi_t(e).$$

Consider the curves $\gamma: t \mapsto e^{tA}$ on $\text{GL}_n(\mathbb{R})$.

By defⁿ of the maps $t \mapsto \exp(t\tilde{A})$,
 we have $\exp(t\tilde{A})$ is the maximal int.
 curve of $\tilde{A}$ through I , while γ satisfies

$$\begin{aligned} \dot{\gamma}(0)(f) &= \frac{d}{dt} \Big|_{t=0} (f \circ \gamma) & (f \in C^\infty(\text{GL}_n)) \\ &= \frac{d}{dt} \Big|_0 f(e^{tA}) = \sum \frac{\partial f}{\partial x_{ij}} (A)_{ij} \\ &= \left(\sum a_{ij} \frac{\partial}{\partial x_{ij}} \Big|_I \right) (f). \\ &= \tilde{A}(I)(f). \end{aligned}$$

Hence γ is the max int. curve of
 $\tilde{A}$ through I . Thus $\exp(tA) = e^{tA}$
 $\forall A \in M_n(\mathbb{R})$. □

By a similar computation (use left-inv.)
 to prove

Lemma:- For a Lie group G , $g \in G$ and
 $X \in \mathfrak{g}$, we have $\text{Exp}_X(g) = L_g(\exp(X))$

Corollary:- For $X \in \mathfrak{g}$, the flow of X
 is given by $F_X(t, g) = g \exp(tX)$.

----- x ----- x ----- x ----- x -----

Lecture - 15

Some recalls $\Rightarrow$ We are studying vector fields on a smooth manifold M . We define a smooth vector field as a map $X: M \rightarrow T(M) = \bigcup_p T_p M$ such that $X(p) \in T_p M \forall p \in M$ and on any local chart (U, ϕ) of M , $X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i}|_q$, $q \in U$; all f_i are smooth wrt any chart.

$\rightarrow$ The set of all smooth vector fields on M is a real vector space $\mathcal{X}(M)$.

$\rightarrow$ For $X, Y \in \mathcal{X}(M)$, (thinking of X as a derivation: $C^\infty(M) \rightarrow C^\infty(M)$; $X(f)(p) := X(p)(f)$)

$[X, Y] := X \circ Y - Y \circ X \in \text{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$ is a vector field, $(\mathcal{X}(M), [\cdot, \cdot])$ is a Lie algebra.

Rem: In general $X \circ Y$ is not a vector field as this may involve higher order diff. operators

• For $X \in \mathcal{X}(M)$, $f \in C^\infty(M)$, $\mathcal{L}_X: C^\infty(M) \rightarrow C^\infty(M)$
 $\equiv \mathcal{L}_X(f)(p) := X(p)(f)$; $\mathcal{L}_X(f)$ is called

the Lie derivative of f wrt X .

We can think of $\mathcal{L}_X(f)$ as the function giving, for $p \in M$, the 'directional derivative' of f

¹⁵⁻² at p , in the direction of $X(p) \in T_p M$.

→ $\mathcal{X}(M)$ is a $C^\infty(M)$ -module. For $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, $(fX)(p) := X(p)(f)$ defines the module structure.

→ $[\cdot, \cdot]: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ is only $\mathbb{R}$ -bilinear, not $C^\infty(M)$ -bilinear.*

→ $\mathcal{L}_X(Y) := [X, Y]$ is the Lie derivative of Y w.r.t X .

$$* [fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X, \\ f, g \in C^\infty(M).$$

→ M, N smooth, $\varphi: M \rightarrow N$ smooth; $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$ are φ -related if $\forall p \in M$, $D\varphi(p)(X(p)) = Y(\varphi(p))$.

• X, Y are φ -related $\Leftrightarrow X(g \circ \varphi) = Y(g) \circ \varphi$
 $\forall g \in C^\infty(N)$.

$$\boxed{M \xrightarrow{\varphi} N, \quad D\varphi(p): T_p M \rightarrow T_{\varphi(p)} N \\ D\varphi(p)(X(p))(g) = X(p)(g \circ \varphi) \\ \equiv Y(\varphi(p)) \quad \forall p.}$$

Lie groups: • Here the important notion is of left inv. vector fields. If G be a Lie grp $X \in \mathcal{X}(G)$ is left-inv. if $D L_g \circ X = X \circ L_g$
 $\forall g \in G$.

• Hence left-inv. vect. fields on G are completely determined by their value at $e \in G$.

^{15.3}
 $\rightarrow$ If X, Y are left inv. vector fields on G , then $[X, Y]$ is left-inv. Hence the subspace $\mathfrak{g}$ of $\mathcal{X}(G)$ is a Lie sub-algebra. The map

$$T_e G \rightarrow \mathfrak{g}, \quad v \mapsto X_v: X_v(g) \in T_g G$$

is given by $X_v(g) := DL_g(e)(v) \in T_g G$.

$\rightarrow$ We identify $\mathfrak{g}$ with $T_e G$ via this isomorphism of $\mathbb{R}$ -vector spaces and give $T_e G$ a Lie alg. structure via this, i.e.

$$[v, w] := [X_v, X_w].$$

• Integral curves, flows of vect. fields :-

Given $X \in \mathcal{X}(M)$ & $p \in M$, $\exists \epsilon > 0$ and a curve $\gamma: (-\epsilon, \epsilon) \rightarrow M$ s.t. $\gamma(0) = p$, $\dot{\gamma}(0) \in T_p M$ and $X(p) = \dot{\gamma}(0)$. We call such a curve an integral curve of X through p , if $\forall t \in (-\epsilon, \epsilon)$, we have, in addition, $X(\gamma(t)) = \dot{\gamma}(t)$.

$\rightarrow$ These exist. An integral curve through p having the largest possible interval of defⁿ is unique & is called the max int curve of X through p , denoted $\gamma_p: I_p \rightarrow M$.

¹⁵⁻⁴ $\rightarrow$ We call X complete if all integral curves of X have domain $= \mathbb{R}$.

- Such fields are important since they generate a global action of $\mathbb{R}$ on M

$$\mathbb{R} \times M \rightarrow M, (t, p) \mapsto \gamma_p(t).$$

The map $\Phi_X: \mathbb{R} \rightarrow \text{Diff}(M)$, $\Phi_X(t) = (p \mapsto \gamma_p(t))$ corresponds to this action.

- Writing φ_t for $\Phi_X(t)$, we see

$$\varphi_s \circ \varphi_t = \varphi_{s+t} \quad \forall s, t \in \mathbb{R}.$$

Flows! Given $X \in \mathcal{X}(M)$, $p \in M$; $\exists \epsilon > 0$ and $p \in U \subseteq M$, U open, a! smooth map

$$F: (-\epsilon, \epsilon) \times U \rightarrow M \text{ with } F(0, x) = x \\ \forall x \in U; \frac{d}{dt}(F(x, t)) = X(F(x, t)) \quad \forall t \in (-\epsilon, \epsilon), x \in U.$$

So the existence & uniqueness theorem gives a! family of integral curves of X on a nbd U of p , having the same domain $I = (-\epsilon, \epsilon)$; $\gamma_p(t) = F(t, p) \neq \varphi_t(p)$.

15.5 • X complete gives $\{\varphi_t\}_{t \in \mathbb{R}}$. This is
 call- $\mathcal{L}$ the 1-param. group of diffeoms assoc.
 to X ; $\varphi_t(p) := \sigma_p(t) = F(t, p)$.

When X is not nec. complete, we still
 get a 1-param. family of local diffeoms:

→ A local 1-param family of local diffeoms
 is a collection $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ s.t.

(i) U_α are open, (ii) $\bigcup_\alpha U_\alpha = M$

(iii) $\epsilon_\alpha > 0 \forall \alpha$; domain $\varphi_t^\alpha \supseteq U_\alpha$ and
 the map $(t, z) \mapsto \varphi_t^\alpha(z)$ is smooth, for
 $z \in U_\alpha, t \in (-\epsilon_\alpha, \epsilon_\alpha)$.

(iv) φ_t^α & φ_t^β agree on $\text{dom}(\varphi_t^\alpha) \cap \text{dom}(\varphi_t^\beta)$
 & t with $|t| < \min\{\epsilon_\alpha, \epsilon_\beta\}$.

(v) for $t, s \in (-\epsilon_\alpha, \epsilon_\alpha)$ with $t+s \in (-\epsilon_\alpha, \epsilon_\alpha)$,

$\varphi_t^\alpha \circ \varphi_s^\alpha$ is defined on an open set $\supseteq U_\alpha$

& $\varphi_t^\alpha \circ \varphi_s^\alpha = \varphi_{t+s}^\alpha$ on U_α .

→ Any local 1-param. family is generated by
 a vector field, i.e. its local flow gives the
 1-param. family; $X(p)(t) = \frac{d}{dt}|_{t=0} f \circ \varphi_t^\alpha, p \in M$.

15-6 Sufficient Condⁿ for completeness: Let $X \in \mathcal{X}(M)$ be s.t. it admits a local 1-param family of diffeoms $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)_\alpha\}$ with $\inf_\alpha (\epsilon_\alpha) > 0$. Then X is complete.

$\Rightarrow$ Vector fields on M are complete if M is compact

• G a Lie group, $X \in \mathfrak{g}$, then X is complete.

Exponential map: Let M be smooth, $X \in \mathcal{X}(M)$ be complete & $\{\varphi_t\}$ be the 1-param group assoc. to X . The exponential map of X is $\text{Exp}_X: M \rightarrow M$, $\text{Exp}_X(p) := \varphi_1(p) = \gamma_p(1)$.

Examples: 1. $A \in M_n(\mathbb{R})$, $A = (a_{ij})$. Let

$$\hat{A} \in \mathcal{X}(\mathbb{R}^n), \quad \hat{A} = \sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i}.$$

Then $\text{Exp}_{\hat{A}}(p) = e^{\hat{A}} \cdot p$ (see lect. 14)

2. If G be a Lie group, any $X \in \mathfrak{g}$ is complete, Exp_X is thus defined. Define

$$\exp(X) := \text{Exp}_X(e) = \varphi_1(e) = \gamma_e(1)$$

$$\exp(tX) = \varphi_t(e); \quad \{\varphi_t\} \text{ is the 1-param}$$

group assoc. to X . The map $\mathfrak{g} \xrightarrow{\exp} G$

15.7 is called the exponential map of G .

$$\rightarrow \text{Exp}_x(g) = L_g(\exp(x)), \quad \forall g \in G. \quad (*)$$

$$\text{For } G = GL_n(\mathbb{R}), \quad \exp(tA) = e^{tA}.$$

$\rightarrow$ For $X \in \mathfrak{g}$, the flow of X is given by $F_x(t, g) = g \exp(tx)$.

- If G is a Lie group, then the map $\mathbb{R} \rightarrow G, t \mapsto \exp(tx)$ is the (!) maximal integral curve of X , passing through e .

Prf! We have, for $g \in G, \varphi_t(g) = \gamma_g'(t)$
 $= F_x(t, g) = g \exp(tx).$

Hence $t \mapsto \exp(tx)$ is the (!) max integral curve of X through e . $\square$

Some properties of $\exp$:

Lemma: $\exp((s+t)x) = \exp(sx) \cdot \exp(tx) \quad \forall s, t \in \mathbb{R}, X \in \mathfrak{g}.$

Proof! For $g \in G, X \in \mathfrak{g}, t \in \mathbb{R}, h \in G$

$$(L_g \text{Exp}_{tx} L_g^{-1})(h) = L_g \text{Exp}_{tx}(\tilde{g}h) = L_g L_{\tilde{g}h}(\exp(tx)) \\ = L_h(\exp tx) = \text{Exp}_{tx}(h)$$

(by (*)). Hence $L_g \text{Exp}_{tx} = \text{Exp}_{tx} L_g$.

15-8

Now

We have, for $g, h \in G$,

$$\exp((s+t)x) =$$

$$(L_g \text{Exp}_{tx} L_g^{-1})(h) = L_g \text{Exp}_{tx}(\bar{g}h) = L_g L_{\bar{g}h}(\exp(tx))$$

by (*).

$$= L_h \exp(tx)$$

$$= \text{Exp}_{tx}(h)$$

$$\therefore L_g \text{Exp}_{tx} = \text{Exp}_{tx} L_g.$$

$$\rightarrow \exp((s+t)x) = \varphi_{s+t}(e) = \varphi_s(\varphi_t(e)). \text{ And}$$

$$\left(\varphi_s(\varphi_t(e)) = \varphi_s\left(\underbrace{L_{\varphi_t(e)}}_{\varphi_t(e)}\right) = \gamma_{\varphi_t(e)}(s) \right)$$

$$= \exp((t+s)x) = \varphi_t(e) \underbrace{\exp(sx)}_{\parallel} \text{ by (*)}$$

$$= \varphi_t(e) \varphi_s(e)$$

$$= \exp(tx) \exp(sx).$$



$\therefore \exp: \mathbb{R} \rightarrow G$ is a Lie group homom.

• Any Lie group map $\varphi: \mathbb{R} \rightarrow G$ is also called a 1-parameter subgroup of G .

$\rightarrow \exp: \mathfrak{g} \rightarrow G$ is a local diffeom around

$0 \in \mathfrak{g}$. [See Kum; 2.8.28, follows from $\exp$ is smooth]

159. Grant $\exp: \mathfrak{g} \rightarrow G$ is smooth. Enough to prove $D(\exp)(0)$ is bijective. Let $X \in \mathfrak{g} \equiv T_e G$. Consider the curve $t \mapsto tX$ in $\mathfrak{g}$. This has tangent X at 0. Consider the image of this curve under $\exp$, i.e. $t \mapsto \exp(tX)$; this has tangent X at e . This shows $D(\exp)(0): \mathfrak{g} \rightarrow T_e G$ is bijective etc. $\square$

→ Let $\varphi: G \rightarrow H$ be a homomorphism of Lie groups. Then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \uparrow \exp & & \uparrow \exp \\ \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \end{array}$$

Commutates, i.e.
 $\exp(d\varphi(X)) = \varphi(\exp(X))$
 $\forall X \in \mathfrak{g}$.

Pf. Consider the curves $t \mapsto \varphi(\exp(tX))$ & $t \mapsto \exp t(d\varphi(X))$. These are both integral curves for $d\varphi(X)$, through $e \therefore$ are equal. $\square$

→ Any one parameter subgroup $\varphi: \mathbb{R} \rightarrow G$ comes from some $X \in \mathfrak{g}$. More precisely, $\exists X \in \mathfrak{g}$ s.t. $\varphi(t) = \text{Exp}_{tX}(e)$.

B. Math (Hons.) III Year 2021-22
II Semester
ATTENDANCE SHEET

Course : Differential Geometry II
Instructor : Maneesh Thakur

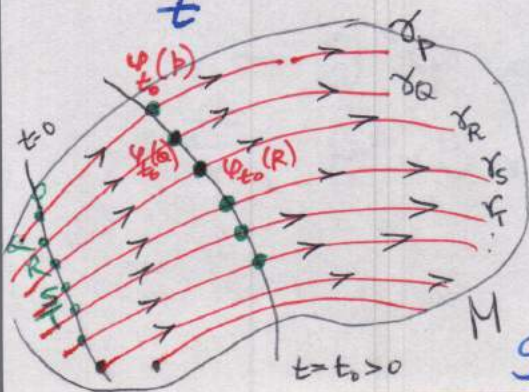
SL.	Name	March												April												Total Class
		Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	
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1	P Anirudha Chakravarthi																									
2	Anupam Nayak																									
3	Arun Vittal Wandakar																									
4	Debabrata Pal																									
5	Harshul Khanna																									
6	Kuntal Das																									
7	Muhammed K M																									
8	Parijat Chakraborty																									
9	Rashmi Konnur																									
10	Rishika Agrawal																									
11	Rohan Raj Padhy																									
12	Safal Das Biswas																									
13	Sayantana Mondal																									
14	Shahbaz Ahmad Khan																									
15	Shubhankar Varshney																									
16	Sneha K K																									
17	Soham Ghosh																									
18	Sonali Priyadarsini Behara																									

Instructor Signature :

Lecture-16

Exponential map: We discussed complete vector fields on manifolds. These are vector fields X on M whose all integral curves (maximal ones) have domain $\mathbb{R}$. If X be such a vector field, F its flow on M ; $F: \mathbb{R} \times M \rightarrow M$ smooth with $\bullet F(0, x) = x \quad \forall x \in M, \bullet \frac{d}{dt} F(t, x) = X(F(t, x)) \quad \forall t \in \mathbb{R}, x \in M$. This gives $\phi: \mathbb{R} \rightarrow \text{Diff}(M)$ a group homomorphism, $\phi(t) =: \varphi_t$, i.e. $\varphi_t(p) = F(t, p)$. $\varphi_s \circ \varphi_t = \varphi_{s+t} \quad \forall s, t \in \mathbb{R}, \varphi_0 = \text{id}_M$. The set $\{\varphi_t\}_{t \in \mathbb{R}}$ is thus a group under $\circ$, $\varphi_t^{-1} = \varphi_{-t}$. Call φ the 1-parameter group of diffeoms assoc. to X . A few remarks are in order:

- The flow of X gives the set $\{\gamma_p\}_{p \in M}$ of max integral curves of X , all having domain $\mathbb{R}$.
- $\varphi_t(p) = F(t, p) = \gamma_p(t)$; $\dot{\gamma}_p(t) = X(\gamma_p(t)) \quad \forall t \in \mathbb{R}$.



- The 1-param group $\{\varphi_t\}$ moves or translates the points of M simultaneously along the max integral curves through those points, as in the picture, giving the action of $\mathbb{R}$ on M .

Now let $X \in \mathcal{X}(M)$ be complete, $\{\varphi_t\}$ the corresp. 1-param group. Define the exponential map of X by $\text{Exp}_X : M \rightarrow M$, $\text{Exp}_X(p) = \varphi_1(p) = \gamma_p(1)$, $p \in M$. So $\text{Exp}_X(p)$ is the point on the max int. curve γ_p at $t=1$. We discuss one case of M in some detail, when we know any left inv. Vector field is complete, $M = \underline{G = \text{a Lie group}}$.

We'll discuss (next Tuesday afternoon)

*Exercise 1. $\text{Exp}_X(g) = L_g(\exp(x))$, where

$\exp(x) := \text{Exp}_X(e) = \varphi_1(e)$ is the map

$\mathfrak{g} \equiv T_e G \rightarrow G$, call it the exponential map of G . ($\exp(tX) := \varphi_t(e)$) = $\gamma_e(t)$

• For $X \in \mathfrak{g}$, the flow F_X of X is given by $F_X(t, \alpha) = \alpha \exp(tX)$ for any $t \in \mathbb{R}$, $\alpha \in G$.

• $t \mapsto \exp(tX)$ is the max integral curve of X through e . (~~*Exercise 2~~)

~~This follows from the following computation~~

~~*Exercise 2: $\text{Exp}_X(e) = \varphi_1(e)$~~



16-3

$$\bullet \exp((s+t)x) = \exp(sx) \exp(tx) \quad \forall s, t \in \mathbb{R}, x \in \mathfrak{g}.$$

This follows from $L_g \text{Exp}_{tx} = \text{Exp}_{tx} L_g$. (*)

$$\exp((s+t)x) = \varphi_{s+t}(e) = \varphi_s(\varphi_t(e))$$

$$= \varphi_t(\varphi_s(e)) = \varphi_t(L_{\varphi_s(e)}(e))$$

$$= L_{\varphi_s(e)}(\varphi_t(e))$$

$$= \varphi_s(e) \varphi_t(e).$$

(Left inv. of $X \equiv DL_g \circ X = X \circ L_g$ is used to have (*)).

Exercise: Fill details in the above.

$\bullet \exp: \mathfrak{g} \rightarrow G$ is a local diffeo around $0 \in \mathfrak{g}$.

To prove this, we think of $\mathfrak{g}$ as a Euclidean space, prove $\bullet \exp$ is smooth: $\mathfrak{g} \rightarrow G$ and observe, for $X \in \mathfrak{g}$, the curve $t \mapsto tX$ in $\mathfrak{g}$ has X as its tangent at 0 & $\exp$ takes this curve to the curve $t \mapsto \exp(tX)$, whose tangent at $t=0$ is $X(e)$.

16-4 We'll need

Lemma: $\phi: G \rightarrow H$ be a homom of Lie groups,

Then

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \exp \uparrow & \curvearrowright & \uparrow \exp \\ \mathfrak{g} & \xrightarrow[\text{D}\phi(e)]{d\phi} & \mathfrak{h} \end{array}$$

$$\begin{aligned} \text{i.e. } \exp(d\phi(x)) \\ = \phi(\exp(x)) \\ \forall x \in \mathfrak{g}. \end{aligned}$$

Pf: Compare the curves $t \mapsto \phi(\exp(tx))$ & $t \mapsto \exp(td\phi(x))$ on H through $e \in H$; both are integral curves of $d\phi(x)$ (?). $\square$

Cor: $d\phi(x) = \frac{d}{dt} \phi(\exp(tx)) \Big|_{t=0}$.

- $\phi, \psi: G \rightarrow H$ be Lie group homoms, G connected and $d\phi = d\psi$. Then $\phi = \psi$.

Proof: We have, for all $X \in \mathfrak{g}$

$$\begin{aligned} \exp(d\phi(x)) &= \phi(\exp(x)) = \exp(d\psi(x)) \\ &= \psi(\exp(x)). \end{aligned}$$

Since $\exp: \mathfrak{g} \rightarrow G$ is a local diffeo, we

conclude that $\phi = \psi$ on an open nbd of $e \in G$.

The result follows from this (?). $\square$

16-5
Exercises: Let G be a topological group, i.e.
 G is Hausdorff, $G \times G \xrightarrow{\mu} G$, $i: G \rightarrow G$ are
 Continuous.

1. Let U be an open set $\ni e$. Then $\exists e \in V \subset U$
 open
 s.t. $V = V^{-1}$ & $VV = VV^{-1} \subseteq U$.

(Such open sets V are called symmetric)

Hint: Use contin. of μ to find opens P, Q both $\ni e$

s.t. $P, Q \subseteq U$, $W = P \cap Q$ & take $V = W \cap W^{-1}$.

2. $G = \bigcup_{k=1}^{\infty} U^k := \{u_1 \dots u_k \mid u_i \in U\}$ if G is conn. & U symmetric.

any open $\ni e$. So U generates G as an
 abstract group.

Frobenius Theorem & applications

Let M be smooth, $X \in \mathcal{X}(M)$ be s.t. $X(p) \neq 0$
 $\forall p \in M$. We have seen then that $\exists$ integral
 curve γ_p of X through p . The condition on
 $X \Rightarrow \dot{\gamma}_p(0) \neq 0$. This implies, for $\epsilon > 0$
 suff. small, $\gamma_p: (-\epsilon, \epsilon) \rightarrow M$ is a 1-1 smooth
 immersion (?).

• The condⁿ $X(p) \neq 0 \forall p \in M$ implies that
 $\mathbb{D}_p := \langle X(p) \rangle \subset T_p M$ is a 1-dim'l subspace $\forall p$.

16-6 We call the association $p \mapsto \mathcal{W}_p$ a 1-dim'l "differential system" on M .

So, the existence of integral curves of X can be rephrased by saying: given a 1-dim'l differential system on M and $p \in M$, $\exists$ a submanifold $S \subseteq M$, $p \in S$, such that $T_q S = \mathcal{W}_q \forall q \in S$; with $S = ((-\epsilon, \epsilon), \gamma_p)$.

More generally, we define a d -dim'l diff system on M , $p \mapsto \mathcal{W}_p \subset T_p M$, $\mathcal{W}_p$ a d dim'l subspace of $T_p M$; we also demand

- $\forall p \in M$, $\exists \overset{\text{open}}{p \in U \subseteq M}$, $X_1, \dots, X_d \in \mathcal{X}(U)$

such that $\mathcal{W}_q = \langle X_1(q), \dots, X_d(q) \rangle \forall q \in U$.

One then says the vector fields $X_1, \dots, X_d$ span $\mathcal{W}$ & call $\mathcal{W}$ smooth.

Question: Given a smooth d -dim'l diff system on M , $p \in M$, $\exists$ a submanifold S passing through p & $T_q S = \mathcal{W}_q \forall q \in S$?

We define a few notions first:

Involutive diff systems:- A diff. system $\mathcal{W}$ is involutive if $X, Y \in \mathcal{W} \Rightarrow [X, Y] \in \mathcal{W}$, i.e. $\rightarrow$

16.7 for any $p \in M$, $[X, Y](p) \in \tilde{W}_p$.

- We call a submanifold S of M an integral submanifold at p for $\tilde{W}$ if $p \in S$ and $\forall q \in S$,

$$T_q S = \tilde{W}_q.$$

- We call $\tilde{W}$ integrable if for every $p \in M$, there is an integral submanifold through p .

Frobenius theorem : A differential system $\tilde{W}$ is integrable $\Leftrightarrow \tilde{W}$ is involutive.

Remarks/example : • Let $\tilde{W}$ be a d -dim'l diff system. Then for $p \in M$, $\exists \underset{p}{U} \subseteq M$ open, $X_1, \dots, X_d \in \mathcal{X}(U)$ s.t. $X_1, \dots, X_d$ span $\tilde{W}$ on U .

To say $\tilde{W}$ is involutive is equivalent to existence of functions c_{ijk} s.t., $\forall i, j$,

$$[X_i, X_j] = \sum_{k=1}^d c_{ijk} X_k.$$

example : The above is ensured for example

When $\exists$ open U , $p \in U$, such that $\exists X_1, \dots, X_d \in \mathcal{X}(U)$ with $[X_i, X_j] = 0 \forall i, j$ & $X_i \in \tilde{W} \forall i$. For a concrete example,

take $X_i = \frac{\partial}{\partial x_i}$ on U

The global Frobenius theorem :- Let $\mathcal{D}$ be a d -dim'l diff. system, involutive, on M & $p \in M$. Then $\exists$! maximal connected integral submanifold of $\mathcal{D}$, passing through p . Every connected integral submanifold of $\mathcal{D}$ through p , is contained in the maximal one.

Application to Lie groups

Thm! Let G be a Lie group, $\mathfrak{g} = \text{Lie } G$ & $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Then $\exists$! connected Lie subgroup $H \leq G$ with $\text{Lie}(H) = \mathfrak{h}$.

Prf! ~~For~~ $g \in G$, ~~and~~ define $\mathcal{D}_g = \{X \in \mathfrak{g} \mid X(g) \in \mathfrak{h}\}$. Then $\mathcal{D}$ is involutive diff system of rank $d = \dim \mathfrak{h}$ for every $g \in G$.

By the Global Frobenius, $\exists$! a maximal integral submanifold $H \ni e$, of $\mathcal{D}$.

Claim! $H \leq G$. To see this, let $h \in H$. Then $L_h(H)$ is a connected integral submfd for $\mathcal{D}$ & $h \in L_h(H)$; Hence $L_h(H) \subseteq H$. This implies $H \leq G$.

→ One can prove further, H is a Lie subgroup. $\square$

Hence we have (Frobenius' thm).

Theorem: G a Lie group, $\mathfrak{g} = \text{Lie } G$. Then
 $\exists$ a bijective correspondence between
Connected Lie subgroups of G and Lie subalgs
of $\mathfrak{g}$. $\square$

Some useful exponential formulae

Thm: Let G be a Lie group, $\mathfrak{g} = \text{Lie } G$;

$X, Y \in \mathfrak{g}$. Then

$$(i) \exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X, Y] + o(t^3)\right)$$

$$(ii) \exp(tX)\exp(tY)\exp(-tX)\exp(-tY) \\ = \exp\left(t^2[X, Y] + o(t^3)\right)$$

$$(iii) \exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X, Y] + o(t^3)\right).$$

Where $o(t^3)$ is a vector in $T_e G$ such that
for $|t| < \epsilon$ $\frac{1}{t^3}o(t^3)$ is bounded & smooth. $\square$

Cartan's theorem! G be a Lie group, $H \leq G$
a closed subgrp. Then $\exists!$ diff. structure on H
so that H is a Lie group & the inclusion map
 $H \hookrightarrow G$ is a Lie homom. $\square$

If $H \leq G$ is a Lie subgroup, then
 $\text{Lie}(H) = \{X \in \mathfrak{g} \mid \exp(tX) \in H \ \forall t \in \mathbb{R}\}.$

The adjoint representation :- Let G be a Lie grp,
 $g, x \in G$, $\iota_g: G \rightarrow G$ be the map $x \mapsto gxg^{-1}$.
 Then ι_g is a Lie group homomorphism.

Notation: Let $\text{Ad}(g) = d\iota_g$ correspond to
 the derivative of $\iota_g: G \rightarrow G$, so $\text{Ad}(g)$
 $: \mathfrak{g} \rightarrow \mathfrak{g}$ is an element of $\text{GL}(\mathfrak{g})$.

Lemma for GL_n : Let $G = \text{GL}_n(\mathbb{R})$, $A \in \text{GL}_n$,
 so $\iota_A(x) = AxA^{-1}$. We have $\text{Ad}(A)(x) = AxA^{-1}$
 $\forall x \in \mathfrak{gl}_n(\mathbb{R}) =: \mathfrak{g}$.

Pf: Recall that $\exp: \mathfrak{gl}_n(\mathbb{R}) \rightarrow \text{GL}_n(\mathbb{R})$ is
 the matrix exponential and $t \mapsto \exp(tX)$
 is the integral curve through I for X .

Hence $\text{Ad}(A)(x) = d\iota_A(x) = \frac{d}{dt} \exp(tAxA^{-1}) \Big|_{t=0}$
 $\left(= \frac{d}{dt} A \exp(tX) A^{-1} \right)$. Since

~~$\exp$ is a local diffeomorphism~~, it follows that

$$\text{Ad}(A)(x) = AxA^{-1}. \quad (*)$$

$\rightarrow \text{Ad}: \text{GL}_n \rightarrow \mathfrak{gl}_n$ is a Lie group homom,
 called the adjoint representⁿ of G on $\mathfrak{g}$.

Lecture - 17

We discuss the exponential map for Lie grps.

• G a Lie group, $\mathfrak{g}$ its Lie algebra, then $\exp: \mathfrak{g} \rightarrow G$ is a local diffeom around 0. This is an extremely useful result for various reasons.

Proposition :- Let $\varphi: G \rightarrow H$ be a map of Lie groups, then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \end{array} \quad \text{Commutates, so}$$

$$\exp(d\varphi(x)) = \varphi(\exp(x)) \quad \forall x \in \mathfrak{g}.$$

Proof :- Consider the curves $t \mapsto \varphi(\exp(tx))$ and $t \mapsto \exp(t d\varphi(x))$ on H . Both these pass through $e \in H$, have domain $\mathbb{R}$, and their derivatives at $t=0$ give $d\varphi(x) \in \mathfrak{h}$, hence they must be equal; thus, at $t=1$, we get $\exp(d\varphi(x)) = \varphi(\exp(x))$. $\square$

17-2. This equality gives, treating $t \mapsto \varphi(\exp(tx))$ as the max curve of $d\varphi(x)$,

$$d\varphi(x) = \left. \frac{d}{dt} \right|_{t=0} \varphi(\exp(tx)).$$

Another important result: applicⁿ of the above,

Proposition: $\varphi, \psi: G \rightarrow H$ be Lie group homomorphisms with $d\varphi = d\psi$. Then $\varphi = \psi$ if we assume G is connected.

Proof: We have, for any $X \in \mathfrak{g}$

$$\exp(d\varphi(x)) = \varphi(\exp(x)) = \exp(d\psi(x))$$

$$\begin{matrix} \mathfrak{h} & \longrightarrow & H \\ \exp & & \end{matrix} = \psi(\exp(x)).$$

Since $\exp: \mathfrak{g} \rightarrow G$ is a local diffeom,

we conclude that $\varphi = \psi$ on a nbd of $e \in H$. The result now follows from

• In a connected topological group G , any nbd of e generates G

• φ & ψ are homomorphisms.



17-3

Discussion on Frobenius Thm, sketch of a proof: Let M be a manifold and $\mathcal{D}$ be a d -dim'l diff. system ($\equiv$ distribution) on M , What does this mean?

• $\mathcal{D}$ assigns to each $p \in M$, a d -dim'l subspace $\mathcal{D}_p$ of $T_p M$

• For every $p \in M$, $\exists U \subseteq M$, $p \in U$ and U open

smooth vector fields $X_1, \dots, X_d \in \mathcal{X}(U)$ such that $\forall q \in U$, $\mathcal{D}_q = \langle X_1(q), \dots, X_d(q) \rangle$ (lin. span)

" $\mathcal{D}$ is smooth"

$\rightarrow$ This generalizes the situation of a vector field on M which has no zeros on M . Given such a vector field, we have integral curves through points of M , that 'fit' the field.

Q: Given smooth $\mathcal{D}$ on M & $p \in M$, $\exists$?

a manifold S & a smooth immersion/embedding

$i: S \rightarrow M$, s.t. $p \in i(S) \leftarrow L_{i(q)}: T_q(S) \rightarrow T_p M$

has image $\mathcal{D}_{i(q)} \in \mathcal{D} \forall q \in S$.

17.4

We then call S an integral (submanifold) of $\mathcal{W}$. We have a nec. condⁿ for existence for such S : first, recall that we say a vector field $X \in \mathcal{W}$ if $X(q) \in \mathcal{W}_q$ for any $q \in M$.

Now if $Y_1, Y_2 \in \mathcal{X}(S)$, then for $i: S \rightarrow M$, $i_*(Y_1), i_*(Y_2) \in \mathcal{W}$,

$$[i_*Y_1, i_*Y_2] = i_*([Y_1, Y_2])$$

This forces: Whenever $X, Y \in \mathcal{X}(M)$ & $X, Y \in \mathcal{W}$, then $[X, Y] \in \mathcal{W}$, if $\mathcal{W}$ were to admit an int. manifold.

Such $\mathcal{W}$ is called integrable.

A non-example + exercise: The 2-dim'l

system spanned by $X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$,

$Y = \frac{\partial}{\partial y}$, is not integrable. Here

X, Y define a 2-dim'l distribution on $\mathbb{R}^3$.

17-5:

We call a distribution $\mathcal{W}$ involutive, if

$$X, Y \in \mathcal{W} \Rightarrow [X, Y] \in \mathcal{W}$$

(Note that $\mathcal{W}$ is not a set of vect. fields!)

Frobenius Thm (local): Let M be a manifold, $p \in M$. Then a d -dim'l distribution is integrable around p ($\equiv \exists$ an integral submanifold passing through p) if and only if $\mathcal{W}$ is involutive.

Let us discuss this. The system on $M: \mathbb{R}^3 - \{x=y=0\}$ spanned by $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial z}$ is involutive, since $[X, Y] = 0$;

Integral curves: For $(x, y, z) \in M$, the integral curve is a circle in the xy -plane, with center at the origin. The integral curve of Y is the line $\parallel$ to the z -axis. The integral manifold through (x, y, z) is the cylinder with z axis as its central axis (Exercise: verify this).

17-6: If $\mathcal{W} = \langle X_1, \dots, X_d \rangle$ on $U \ni p$,
then $\forall i, j$ we can write

$$[X_i, X_j] = \sum_{k=1}^d C_{ijk} X_k, \quad C_{ijk} \text{ smooth on } U,$$

assuming $\mathcal{W}$ is involutive. One obvious
case of this is when all X_i 's commute
i.e. $C_{ijk} = 0 \quad \forall i, j, k$, so $[X_i, X_j] = 0$
 $\forall i, j$. There are a couple of technical
lemmas that go into helping us:

1. Let $X \in \mathcal{X}(M)$, $p \in M$ with $X(p) \neq 0$.
Then $\exists$ a coordinate chart $U \ni p$ such
that $X = \frac{\partial}{\partial x_1}$ on U . (Thm 2.9.2, Kum)

2. $X_1, \dots, X_d \in \mathcal{X}(M)$, $p \in M$, be lin. indep
on $U \ni p$ (i.e. $\langle X_1(q), \dots, X_d(q) \rangle$ is
 d -dim'l $\forall q \in U$). Assume $[X_i, X_j] = 0$
 $\forall i, j$. Then $\exists$ a coord chart $V \ni p$
s.t. $X_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq d$, on V . $\square$

So, by (2), $\exists$ local coordinates s.t.
 $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, d$; $x_i(p) = 0$, $1 \leq i \leq m = \dim M$

17-7

The submanifold S through p ~~has~~ should have

$$\langle X_1(p), \dots, X_d(p) \rangle = T_p S = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right\rangle.$$

S is the 'slice'

$$\{q \in U \mid x_j(q) = 0, d+1 \leq j \leq m\}$$

and for any $q \in U$, the integral submanifold of $\mathcal{W}$ through q is given by the slice

$$\{z \in U \mid x_j(z) = x_j(q), j \geq d+1\}.$$

The general case is reduced to the above:

Write $X_i = \sum_j f_{ij} \frac{\partial}{\partial x_j}$, $f_{ij} \in C^0(U)$. The matrix $(f_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq m}}$ then has rank d .

Wlog can assume $(f_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d}}$ is invertible in a smaller nbhd $V \subseteq U$ of p . Let $(g_{ij})_{1 \leq i, j \leq d}$ be its inverse. Define

$$Y_i = \sum_j g_{ij} X_j. \text{ Then}$$

$$Y_i \in \mathcal{D} \quad \forall i, \quad \langle Y_1, \dots, Y_d \rangle = \mathcal{D}, \quad [Y_i, Y_j] = 0 \quad \forall i, j.$$



Lecture - 18

Discussion on Frobenius Theorem :- We'll discuss mainly examples & notions. Let M be an m -dim'l manifold. We then saw that any smooth vector field X on M can be 'integrated' locally around any point, to an integral curve. • If $\{X(p) = 0\}$, then the corresp integral curve is the constant curve $\gamma_p(t) = p \forall t$. While $X(p) \neq 0$, then the integral curve $\gamma_p(t)$ is a 1-dim'l curve near p .

Higher dim'l analogue :- A d -dim'l distribution $\tilde{W}$ on M is a map (?) that assigns to every $p \in M$ a d -dim'l subspace $\tilde{W}_p$ of $T_p M$. We call $\tilde{W}$ smooth if for every $p \in M$, $\exists$ a nbd U of p & $X_1, \dots, X_d \in \mathcal{X}(U)$ s.t. $\forall q \in U$, $\{X_1(q), \dots, X_d(q)\}$ is a basis of $\tilde{W}_q$; (in particular, $X_1(q) \neq 0, \dots, X_d(q) \neq 0 \forall q \in U$).

We'll discuss only smooth distributions.
• We say a vector field X belongs to $\tilde{W}$ if

18-2

$$X(p) \in \tilde{W}_p \quad \forall p \in M.$$

(in the language of bundles, $\tilde{W}$ is a rank- d subbundle of TM).

Integral manifolds:- Let $\tilde{W}$ be a d -dim'l distribution ($\equiv$ diff. system) on M . A natural question is: does $\exists$ a manifold S and a smooth immersion $\iota: S \rightarrow M$ s.t. $p \in \iota(S)$ and $\iota_* = D\iota(q): T_q(S) \rightarrow T_{\iota(q)}M$ has image $\tilde{W}_{\iota(q)} \quad \forall q \in S$.

If such (S, ι) exists, we call it an integral manifold of $\tilde{W}$. We call $\tilde{W}$ integrable if through each $p \in M$, $\tilde{W}$ admits an integral manifold.

Examples:- 1. Let X be a non-vanishing vector field on M . Then $p \mapsto \langle X(p) \rangle$ is clearly a 1-dim'l distribution on M . Any integral curve of X is an integral manifold of $\tilde{W}$.

2. The vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$

18-3
span a k -dim'l (smooth) distribution $\mathcal{W}$ on $\mathbb{R}^n$.
The integral manifolds of $\mathcal{W}$ are the
'planes' defined by the system of eq's

$$x_j = c_j, \quad k+1 \leq j \leq n.$$

3. Let $M = S' \times S' \subset (\mathbb{R}^2, \{x\}) \times (\mathbb{R}^2, \{y\})$ and
 c be a fixed irrational $\in \mathbb{R}$. The non-
vanishing vector field

$$X = \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right) + c \left(y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \right)$$

defines a 1-dim'l distribution on M ,
the integral manifold is a dense curve
in M !

Remark: Though every 1-dim'l distribution
is integrable, the same fails to be true
for higher dim'l distributions.

Example:- Consider the 2-dim'l distribution
on $\mathbb{R}^3$ defined by (spanned by) the
vector fields $X_1 := \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$ and

$$X_2 := \frac{\partial}{\partial y}, \quad \{x, y, z\} \text{ usual coords on } \mathbb{R}^3.$$

13-4 Then $\mathcal{W}$ is not integrable. In fact, there is no integral manifold through $(0,0,0)$.
(should discuss this).

Observe that $[X_1, X_2] = -\frac{\partial}{\partial z} \notin \mathcal{W}$.

• Let us discuss conditions for integrability:
If (S, ι) is an integral manifold for $\mathcal{W}$,
and $Y_1, Y_2 \in \mathcal{X}(S)$, then $\iota_*(Y_1), \iota_*(Y_2) \in \mathcal{W}$ (?); $[Y_1, Y_2] \in \mathcal{X}(S)$, so

$$\iota_*([Y_1, Y_2]) = [\iota_*(Y_1), \iota_*(Y_2)] \in \mathcal{W}.$$

This therefore gives us a nec. condⁿ for $\mathcal{W}$ to admit an integrable manifold:

• We call a distribution $\mathcal{W}$ involutive if
 $X, Y \in \mathcal{W} \Rightarrow [X, Y] \in \mathcal{W}$.

• The above example is not involutive,
hence not integrable!

Examples (Involutive dist.) • Any 1-dim'l
distribution is involutive, since $[fX, gX]$
 $= f(X(g))X - g(X(f))X$.

18-5 • Let $f: M \rightarrow N$ be a submersion. Define the distribution $\mathcal{W}$ by $\mathcal{W}_p = \text{Ker}(Df(p)) \subseteq T_p M$. If X, Y be vector fields belonging to $\mathcal{W}$, then $Df(p)(X(p)) = Df(p)(Y(p)) = 0$. Hence both X, Y are f -related to $0 \in \mathcal{X}(N)$. Clearly $Df(p)([X, Y](p)) = 0$. Hence $\mathcal{W}$ is involutive.
 $\rightarrow$ The integral manifold passing through $p \in M$ is the submanifold $f^{-1}(f(p))$.

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 Take up Lecture-17, page-3 onwards from here.

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Lecture-19

- Discuss proof of local Frobenius (Lect-17)
- Global Frobenius w/o proof.
- Applicⁿ to Lie groups - Lie alg. Correspondence.

Lie groups (contd): We briefly discuss the adjoint repⁿ of a Lie group on its Lie alg. The general philosophy is that 'to study a group, one must write down all its repⁿs'. The adjoint repⁿ is natural to the structure of the Lie group and is realized concretely as a repⁿ. We prove, for the conjugation

$\iota_g: G \rightarrow G$, $D(\iota_g)(e): \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra autom, so $D(\iota_g)(e) =: \text{Ad}(g) \in \text{GL}(\mathfrak{g})$, and we get the adjoint repⁿ $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$. When $G = \text{GL}_n$, $A \in \text{GL}_n$, then $\text{Ad}(A)(x) = AxA^{-1}$, $\forall x \in \mathfrak{gl}_n = \text{Lie}(\text{GL}_n)$. We have,

$\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ is smooth, hence $D(\text{Ad})(e): \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra map. We now compute this.

13-2

Example : $G = GL_n$, $\mathfrak{g} = \mathfrak{gl}_n = M_n(\mathbb{R})$.

Denoting $DAd(e) =: ad$, we claim

$ad: \mathfrak{g} \rightarrow \mathfrak{gl}_n$ is given by

$$ad(x)(y) := xy - yx = [x, y].$$

We are looking to differentiate $Ad: GL_n \rightarrow GL(\mathfrak{gl}_n)$.

Using the commutative diag

$$\begin{array}{ccc} GL_n & \xrightarrow{Ad} & GL(\mathfrak{gl}_n) \\ \uparrow \exp & \cup & \uparrow \exp \\ \mathfrak{gl}_n & \xrightarrow{ad} & \mathfrak{gl}(\mathfrak{gl}_n) \end{array} \quad \text{we get,}$$

using $\exp(x) = e^x$ for any matrix X ,

$$e^{ad(x)} = Ad(e^x). \quad \text{For } t \in \mathbb{R}, \text{ this gives}$$

$$e^{ad(tx)} = Ad(e^{tx}). \quad \text{Hence, for any } y \in \mathfrak{gl}_n,$$

$$\cancel{e^{ad(tx)} \cdot (ty) e^{-ad(tx)}} = \text{we get}$$

$$\begin{aligned} & (I + ad(tx) + ad(tx)^2 + \dots)(ty) \\ &= e^{tx} \cdot ty \cdot e^{-tx} \end{aligned}$$

$$LHS = ty + t^2 ad(x)(y) + t^3 ad(x)^2(y) + \dots$$

19-3
RHS

$$= \left(1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots\right) tY \left(1 - tX + \frac{t^2 X^2}{2!} - \frac{t^3 X^3}{3!} + \dots\right)$$

$$= \left(tY + t^2 XY + \frac{t^3 X^2 Y}{2!} + \frac{t^4 X^3 Y}{3!} + \dots\right) \left(1 - tX + \frac{t^2 X^2}{2!} - \frac{t^3 X^3}{3!} + \dots\right)$$

$$= \underset{\substack{\uparrow \\ \text{deg 1 int}}}{tY} + \underset{\substack{\uparrow \\ \text{deg 2 int}}}{(-t^2 YX + t^2 XY)} + \underset{\substack{\uparrow \\ \text{deg} \geq 3 \text{ int}}}{\left(-\frac{t^3 X^2 Y}{2!} + \frac{t^4 X^3 Y}{3!} + \dots\right)}$$

Comparing the two, we get

$$\text{ad}(X)(Y) = XY - YX = [X, Y].$$

The general case is similar:

$$\begin{array}{ccc} G & \xrightarrow{\text{Ad}} & GL(\mathfrak{g}) \\ \uparrow \text{exp} & \uparrow \text{exp} & \\ \mathfrak{g} & \xrightarrow{\text{ad}} & gl(\mathfrak{g}) \end{array} \Rightarrow \begin{aligned} &\exp(\text{ad}(X)) \\ &= \text{Ad}(\exp(X)) \\ &\forall X \in \mathfrak{g}. \end{aligned}$$

So, for $t \in \mathbb{R}$ (small) $\forall Y \in \mathfrak{g}$,

$$\exp(\text{ad}(tX))(tY) = \text{Ad}(\exp(tX))(tY)$$

$$\begin{aligned} &\stackrel{||}{=} e^{\text{ad}(tX)}(tY) = \left(I + t \text{ad}(X) + \frac{t^2 \text{ad}(X)^2}{2!} + \dots\right)(tY) \\ &= tY + t^2 \text{ad}(X)(Y) + t^3 \left(\frac{\text{ad}(X)^2(Y)}{2!}\right) + \dots \end{aligned}$$

19-4

On the other hand

$$\exp(\operatorname{Ad}(\exp(tx))(ty))$$

||

$$\begin{aligned} \exp\left(D_{\exp(tx)}(e)(ty)\right) &= \cancel{\exp} \\ &= \exp(tx) \exp(ty) \exp(-tx) \\ &= \exp(ty + t^2[x, y] + o(t^3)) \end{aligned}$$

By the exponential conj. formula

$$= \exp(ty + t^2[x, y] + o(t^3))$$

Hence, for t suff small, we conclude

$$\operatorname{Ad}(\exp(tx))(ty) = ty + t^2[x, y] + o(t^3)$$

$$= ty + t^2 \operatorname{ad}(x)(y) + o(t^3)$$

$$\therefore \operatorname{ad}(x)(y) = [x, y] \quad \forall x, y \in \mathfrak{g}. \quad \square$$

Defⁿ: Let $\mathfrak{g}$ be a Lie algebra. An ideal of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{d} \subseteq \mathfrak{g}$ such that $\operatorname{ad}(x)(\mathfrak{d}) \subseteq \mathfrak{d} \quad \forall x \in \mathfrak{g}$.

Example :- $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$ is an ideal of

$\mathfrak{gl}_n$; $\mathfrak{sl}_n = \{x \mid \operatorname{tr}(x) = 0\} \subseteq \mathfrak{gl}_n$ is an ideal.

19-5 We saw that Lie subalgebras of $\mathfrak{g}$ correspond to connected Lie subgroups of G . What do ideals correspond to?

Thm:- Let H be a connected Lie subgroup of G and $\mathfrak{h}$ be a normal subalgebra. Then $\mathfrak{h} = \text{Lie}(H)$ is an ideal in $\mathfrak{g} = \text{Lie } G$.

* Converse: If G be connected, $\mathfrak{h} = \text{Lie } H$ is an ideal, then H is necessarily normal.

Pf:- Let $H \trianglelefteq G$. Let $Y \in \mathfrak{h}$ & $X \in \mathfrak{g}$. Then $\exp(tY) \in H \forall t \in \mathbb{R}$. Using normality, we have, for $s, t \in \mathbb{R}$

$$\exp(sX) \exp(tY) \exp(-sX) \in H$$

• For any $g \in G$, we consider the $\square$ diag.

$$\begin{array}{ccc} G & \xrightarrow{g} & G \\ \uparrow \exp & & \uparrow \exp \\ \mathfrak{g} & \xrightarrow{\text{Ad}(g)} & \mathfrak{g} \end{array} \quad \begin{array}{l} \text{to get } g \exp(tY) g^{-1} \\ = \exp(t \text{Ad}(g)(Y)). \end{array}$$

Since $\exp$ is a local diffeo: $\mathfrak{g} \rightarrow G$,
 $\rightarrow$ we deduce that $\exp(t \text{Ad}(g)(Y)) \in H$.

• The curve $\gamma(t) = \exp(t \text{Ad}(g)(Y))$ is a 1-param subgroup of H corresp. to the left invariant vector field $\text{Ad}(g)(Y)$ on H .

19-6 For $g = \exp(sx)$,

$$\text{Ad}(\exp(sx))(Y) = e^{s \text{ad}(x)}(Y) \in \mathfrak{h}$$

$\forall s \in \mathbb{R}$, by considering

$$\begin{array}{ccc} G & \xrightarrow{\text{Ad}} & \text{GL}(\mathfrak{g}) \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{g} & \xrightarrow{\text{ad}} & \text{gl}(\mathfrak{g}) \end{array}$$

Note now that the curve

$s \mapsto e^{s \text{ad}(x)}(Y)$ has its tangent $\text{ad}(x)(Y)$ on $\mathfrak{h}$ at 0. Hence $[X, Y] \in \mathfrak{h}$.

* Converse: discussion hour.

→ G be connected. Then $\text{Ker}(\text{Ad}) = Z(G)$.

→ $Z(G)$ is a closed Lie Subgroup &

• $\text{Lie}(Z(G)) = \{X \in \mathfrak{g} \mid \text{ad}(X)(Y) = 0 \forall Y \in \mathfrak{g}\}$.

* $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. Then $\text{ad}(\mathfrak{g})(\mathfrak{h}) \subseteq \mathfrak{h}$. Hence, for $P = \text{Ad}_{D_P(e)} = \text{ad}$ satisfies $D_P(e)(X)(\mathfrak{h}) \subseteq \mathfrak{h}$.

Exercise: 1. Let $P: G \rightarrow \text{GL}(V)$ be a repⁿ of Lie groups; $W \subseteq V$ subspace s.t. $P(g)(W) \subseteq W \forall g \in G$. We call W a G-inv. subspace. Prove for a G-inv. subspace W , we have $D_P(e)(X)(W) \subseteq W \forall X \in \mathfrak{g} = \text{Lie } G$.

2. G be connected, $P: G \rightarrow \text{GL}(V)$ as above, $W \subseteq V$ a $\mathfrak{g}$ -inv. Subsp, i.e. $D_P(e)(X)(W) \subseteq W \forall X \in \mathfrak{g}$. Then W is G-inv.

→ Hence (2) $\Rightarrow P(g)W \subseteq W$ i.e. $\text{Ad}(g)W \subseteq W$. This $\Rightarrow g \exp(W)g^{-1} = g(\exp(W)) = \exp(d_g(e)(W)) = \exp(\text{Ad}(g)W) \subseteq \exp(W)$; $\langle \exp(W) \rangle$ is conn $\Rightarrow H \triangleleft G$.

Lecture-20

Multi-linear algebra / tensors :- Discussion of

higher order derivatives of smooth maps between manifolds naturally leads to the notion of tensors.

- For example if $U \subseteq \mathbb{R}^m$ be open & $\varphi: U \rightarrow \mathbb{R}^n$ a smooth map, we have, for $p \in U$, $D\varphi(p): T_p U \rightarrow T_{\varphi(p)} \mathbb{R}^n$ is an $\mathbb{R}$ -linear map. Identifying $T_p U$ with $\mathbb{R}^m$, $T_{\varphi(p)}(\mathbb{R}^n) = \mathbb{R}^n$, we thus get-

$D\varphi: U \rightarrow \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$, $p \mapsto D\varphi(p)$. This is smooth, hence, for $p \in U$, $D^2\varphi(p): T_p U \rightarrow T_{\varphi(p)} \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n) \cong M_{n \times m}(\mathbb{R})$ is $\mathbb{R}$ -linear. So

$D^2\varphi: U \rightarrow \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n))$ and so on.

We'll study spaces like $\text{Hom}_{\mathbb{R}}(V, \text{Hom}_{\mathbb{R}}(V, W))$ and relate them to 'tensors'.

- The dual: For a (finite dim'l) vector space V over a field K , the dual space V^* is defined by $V^* = \text{Hom}_K(V, K) = \text{all } K\text{-linear maps } V \rightarrow K$. Elements of V^* are called linear forms/functionals. Given a basis of V , $\dim_K V = n < \infty$, say $\{e_1, \dots, e_n\}$, the functionals $\{e_1^*, \dots, e_n^*\}$

²⁰⁻² form the dual basis, $e_i^*(e_j) := \delta_{ij}$, $f = \sum f(e_i) e_i^*$
 for any $f \in V^*$. So $\dim V^* = \dim V = \dim V^{**}$;
 $V \cong V^{**}$ via $v \mapsto (f \mapsto f(v))$.

Multi-linear maps :- V be a vector sp / k . An r -linear ~~map~~ ^{form} on V is a map $\varphi: V \times \dots \times V \rightarrow k$, linear in each variable.

Example :- Let $f_1, \dots, f_r \in V^*$ and define
 $\varphi: V \times \dots \times V \rightarrow k$ by $\varphi(v_1, \dots, v_r) = \prod_{i=1}^r f_i(v_i)$.
 Then φ is r -linear. We denote this r -linear form on V by $\varphi = f_1 \otimes f_2 \otimes \dots \otimes f_r$ (f_1 tensor $f_2 \dots$)
 $\rightarrow$ Let $L^r(V, k) =$ vector space of all r -linear forms on V . We have

Lemma :- For $\varphi \in L^r(V, k)$, $\varphi = \sum \varphi(e_{i_1}, \dots, e_{i_r}) e_{i_1}^* \otimes \dots \otimes e_{i_r}^*$.

$\{e_{i_1}^* \otimes \dots \otimes e_{i_r}^*; 1 \leq i_1, \dots, i_r \leq n\}$ is a basis for $L^r(V, k)$.

Pf : Exercise!

• $\dim_k L^r(V, k) = n^r$. The space $L^r(V, k)$

is called the r -fold tensor product of V^* , denoted by $V^{* \otimes r} := \underbrace{V^* \otimes \dots \otimes V^*}_r = T^r(V^*)$.

More generally, for $V_1, \dots, V_r$ vector spaces, we define $V_1^* \otimes \dots \otimes V_r^*$ as

²⁰⁻³ the space of r -linear forms $V_1 \times \dots \times V_r \rightarrow k$
!!

$$L(V_1, \dots, V_r; k) \equiv V_1^* \otimes \dots \otimes V_r^*. \text{ The}$$

$$\text{forms } \left\{ e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes \dots \otimes e_{i_r}^{r*} \mid \begin{array}{l} 1 \leq i_1 \leq \dim V_1 \\ 1 \leq i_2 \leq \dim V_2 \\ \vdots \\ 1 \leq i_r \leq \dim V_r \end{array} \right\}$$

is a basis of $V_1^* \otimes \dots \otimes V_r^*$.

Pf: Exercise.

For $V_1, \dots, V_r$ finite dim'l k -vect spaces,
we define $V_1 \otimes \dots \otimes V_r = (V_1^*)^* \otimes \dots \otimes (V_r^*)^*$

A $v_1 \otimes \dots \otimes v_r$ is given by

$$(v_1 \otimes \dots \otimes v_r)(f_1, \dots, f_r) = \prod_{i=1}^r f_i(v_i).$$

• Given $T \in \text{Hom}_k(V, W) \exists! T^{\otimes r} \in \text{Hom}(V^{\otimes r}, W^{\otimes r})$

such that $T^{\otimes r}(v_1 \otimes \dots \otimes v_r) = T(v_1) \otimes \dots \otimes T(v_r)$

• In $V \otimes W$, we have

$$v \otimes (\omega_1 + \omega_2) = v \otimes \omega_1 + v \otimes \omega_2$$

$$(v_1 + v_2) \otimes \omega = v_1 \otimes \omega + v_2 \otimes \omega$$

$$\alpha v \otimes \omega = v \otimes \alpha \omega = \alpha(v \otimes \omega), \alpha \in k$$

$$v \otimes (\omega \otimes z) = (v \otimes \omega) \otimes z \quad \text{in } V \otimes W \otimes Z.$$

Alternating forms: Let $\text{char}(k) \neq 2$, V a k -vector space. An r -linear form $\varphi: \underbrace{V \times \dots \times V}_r \rightarrow k$ s.t.

$$\varphi(v_1, \dots, v_i, \dots, v_j, \dots, v_r)$$

$$= -\varphi(v_1, \dots, v_{i-1}, v_j, \dots, v_{j-1}, v_i, \dots, v_r)$$

(obtained by flipping i & j).

Let $\Lambda^r(V) = \{ \varphi \in T^{\otimes r}(V) \mid \varphi: V^* \times \dots \times V^* \rightarrow k \text{ is alternating} \}$.

$$\subseteq T^{\otimes r}(V).$$

We have a natural map $T^r(V) \xrightarrow{\text{Alt}} \Lambda^r(V)$ obtained by 'averaging'

$$\psi \mapsto \left((v_1, \dots, v_r) \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} \text{Sign}(\sigma) \cdot \psi(v_{\sigma(1)}, \dots, v_{\sigma(r)}) \right)$$

Then

• Alt is k -linear, if $\psi \in \Lambda^r(V)$ then

$$\text{Alt}(\psi) = \psi, \text{ for } \psi \in T^r(V), \text{Alt}(\text{Alt}(\psi)) = \text{Alt}(\psi).$$

→ Wedge product :- We have a normalized

$$\text{product } \Lambda^r(V) \times \Lambda^s(V) \rightarrow \Lambda^{r+s}(V)$$

$$(\phi, \psi) \mapsto c \cdot \text{Alt}(\phi \otimes \psi)$$

Where $c = \frac{(r+s)!}{r!s!}$, denoted by $\phi \wedge \psi$.

20-5 • $V \otimes W$ is naturally isom to $\text{Hom}_K(V^*, W)$.

The map $V \otimes W \rightarrow \text{Hom}_K(V^*, W)$ such that $v \otimes w \mapsto (f \mapsto f(v)w)$ gives an isom. Here V, W are both assumed finite dim'l.

Cor! $V^* \otimes V \cong \text{Hom}_K(V, V) = \text{End}_K(V)$.

Symmetric/skew-symmetric = alternating tensors:

Let $t \in V^{\otimes n}$ & $\sigma \in S_n$. We define

$\sigma(t)$ by $\sigma(t)(f_1, \dots, f_n) = t(f_{\sigma(1)}, \dots, f_{\sigma(n)})$,

$f_i \in V^*$. We call t symmetric if $\sigma(t) = t$ $\forall \sigma \in S_n$. Let $\text{Sym}^n(V) =$ all symmetric tensors.

Symmetrization map: $V^{\otimes n} \rightarrow \text{Sym}^n(V)$

$t \mapsto \text{Sym}^n(t)$, where

$$\text{Sym}^n(t)(f_1, \dots, f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} t(f_{\sigma(1)}, \dots, f_{\sigma(n)}).$$

Properties of $\wedge$:- $(\alpha\varphi_1 + \beta\varphi_2) \wedge \psi = \alpha(\varphi_1 \wedge \psi) + \beta(\varphi_2 \wedge \psi)$

$$\varphi \wedge (\alpha\psi_1 + \beta\psi_2) = \alpha\varphi \wedge \psi_1 + \beta\varphi \wedge \psi_2$$

$$\varphi \wedge (\psi \wedge \eta) = (\varphi \wedge \psi) \wedge \eta = \frac{(k+l+r)}{k!l!r!} \text{Alt}(\varphi \otimes \psi \otimes \eta)$$

$$\wedge^k V \otimes \wedge^l V \otimes \wedge^r V ; \varphi \wedge \psi = (-1)^{kl} \psi \wedge \varphi.$$

Thm: Let $\{e_1, \dots, e_n\}$ be a basis of V .

Then $\{e_{i_1} \wedge \dots \wedge e_{i_r}, 1 \leq i_1 < i_2 < \dots < i_r \leq n\}$ is a basis of $\wedge^r V$. Hence $\dim_k \wedge^r V = {}^n C_r$.

Cor: $\dim \wedge^n V = 1$, $n = \dim V$.

Prf: Let $\varphi \in \wedge^r(V)$. Then $\varphi \in \otimes^r V$, so

$$\varphi = \sum_{i_1, \dots, i_r} a_{i_1, \dots, i_r} e_{i_1} \otimes \dots \otimes e_{i_r}. \text{ Since } \varphi \in \wedge^r V,$$

$$\begin{aligned} \varphi &= \text{Alt}(\varphi) = \sum a_{i_1 i_2 \dots i_r} \text{Alt}(e_{i_1} \otimes \dots \otimes e_{i_r}) \\ &= \sum a'_{i_1 i_2 \dots i_r} e_{i_1} \wedge \dots \wedge e_{i_r}, \quad a'_* = \text{mult of } a_* \end{aligned}$$

If $\sum \beta_{i_1, \dots, i_r} e_{i_1} \wedge \dots \wedge e_{i_r} = 0$, then, for a fixed

$$(j_1, \dots, j_r), \quad \sum \beta_{i_1, \dots, i_r} e_{i_1} \wedge \dots \wedge e_{i_r} (e_{j_1}^*, \dots, e_{j_r}^*) = 0$$

$$\Downarrow$$

$$\sum \beta_{i_1, \dots, i_r} e_{i_1}(e_{j_1}^*) \dots e_{i_r}(e_{j_r}^*) = 0$$

$$\therefore \beta_{j_1, \dots, j_r} = 0 \quad \text{etc.} \quad \square$$

• Let $T \in \text{End}(V)$. Then $\exists!$ linear map

$$\wedge^r T: \wedge^r V \rightarrow \wedge^r V \text{ such that}$$

$$\wedge^r(T)(v_1 \wedge \dots \wedge v_r) = Tv_1 \wedge \dots \wedge Tv_r.$$

20-7

Cor! Since $\dim \wedge^n V = 1$, $n = \dim V$, for $T \in \text{End}(V)$, $\wedge^n T : \wedge^n V \rightarrow \wedge^n V$ must be a scalar mult. In fact $\wedge^n T = \det(T)$.

Pf! Let $\{e_1, \dots, e_n\}$ be a basis of V and

$$Te_j = \sum_j a_{ij} e_j. \quad \text{Then } \wedge^n T(e_1 \wedge \dots \wedge e_n)$$

$$= Te_1 \wedge \dots \wedge Te_n = \left(\sum_i a_{i1} e_i \right) \wedge \left(\sum_i a_{i2} e_i \right) \wedge \dots \wedge \left(\sum_i a_{in} e_i \right)$$

$$= \sum_{j_1 \neq j_2 \neq \dots \neq j_n} a_{j_1 1} a_{j_2 2} \dots a_{j_n n} e_{j_1} \wedge \dots \wedge e_{j_n}$$

$$= \sum a_{j_1 1}^* \dots a_{j_n n} \text{sign}(j_1, \dots, j_n) e_1 \wedge \dots \wedge e_n$$

$$= \det(T) e_1 \wedge \dots \wedge e_n.$$



Lecture-21

Differential forms on manifolds :- These are certain objects associated to a manifold for which we can define a notion of integral, use them to define orientations, among other applications.

Let M be a manifold. Then $C^\infty(M)$ is a commutative $\mathbb{R}$ -algebra and $\mathcal{X}(M)$ is a $C^\infty(M)$ -module via $(f \cdot X)(p) = f(p)X(p)$. We denote $\mathcal{X}(M)$ by $D_1(M)$ when treated as a $C^\infty(M)$ -module. Let $D^1(M)$ denote the $C^\infty(M)$ dual of $D_1(M)$, i.e.

$$D^1(M) := \text{Hom}_{C^\infty(M)}(D_1(M), C^\infty(M)), \text{ the}$$

$C^\infty(M)$ -module of $C^\infty(M)$ -linear forms on $D_1(M)$. We call elements of $D^1(M)$ as differential 1-forms on M .

Example :- Let $f \in C^\infty(M)$. Define

$$df : D_1(M) \rightarrow C^\infty(M) \text{ by } df(X) = X(f).$$



²¹⁻² Then $df \in \text{Hom}_{C^\infty(M)}(D_1(M), C^\infty(M)) =: D^1(M)$, and

$\omega = df$ is a differential 1-form. We'll discuss this further (we have done this before!)

Recall the co-tangent space of M is $T_p^*(M) =$ vector space dual of $T_p M$. For $f \in C_p^\infty(M)$, we can define $df(p): T_p M \rightarrow \mathbb{R}$ by

$$df(p)(v) := v(f). \text{ So } df(p) \in T_p^* M.$$

Let $\{x_i\}$ be local coordinates around p .

Then $x_i: U \rightarrow \mathbb{R}$ are smooth and $\{dx_i(p)\}$ is a basis of $T_p^* M$, this is

the dual basis corresponding to the basis $\{\frac{\partial}{\partial x_i}|_p\}$ of $T_p M$. For $v \in T_p M$ &

$$f \in \cancel{C_p^\infty(M)}, \quad v = \sum v(x_i) \frac{\partial}{\partial x_i}|_p. \text{ Hence}$$

$$df(p)(v) = v(f) = \sum v(x_i) \frac{\partial f}{\partial x_i}|_p$$

$$= \sum \frac{\partial f}{\partial x_i}|_p v(x_i)$$

$$= \sum \frac{\partial f}{\partial x_i}|_p dx_i|_p(v). \text{ Hence}$$

$$df(p) = \sum \frac{\partial f}{\partial x_i}|_p dx_i|_p.$$

21-3

→ For $\omega \in \mathcal{D}^1(M) = \text{Hom}_{C^\infty(M)}(\mathcal{D}_1(M), C^\infty(M))$,

we wish to get a local expression.

We proceed to do this now. For $p \in M$,

define $\omega_p \in T_p^*M$ by $\omega_p(v) = \omega(X)(p)$

where $v \in T_p M$ and X is a smooth vector field on M with $X(p) = v$.

ω_p is well defined :- We have seen before

that if $X=0$ on an open set $V \subseteq M$, then $\omega(X)=0$ on V . For this, take $p \in V$ and f a smooth function in a neighborhood of p s.t. $f=0$ on that neighborhood of p , $f=1$ outside V . Then

$\boxed{fX = X}$. Since ω is $C^\infty(M)$ -linear, we have $\omega(fX) = f\omega(X)$. So $\omega(X)(p) = \omega(fX)(p) = f(p)\omega(X)(p) = 0$.

Hence $X=0$ on $V \Rightarrow \omega(X)=0$ on V .

Thus ω induces a 1-form on an open subset U of M . If $X = \sum_i X(x_i) \frac{\partial}{\partial x_i}$ on U ,

then $X(p)=0 \Leftrightarrow X(x_i)(p)=0 \forall i$.

21-4

Hence

$$\omega(x)(p) = \left[\sum_i x(\alpha_i) \omega\left(\frac{\partial}{\partial \alpha_i}\right) \right](p)$$

$$= \sum_i x(\alpha_i) \underbrace{\omega\left(\frac{\partial}{\partial \alpha_i}\right)(p)}_0 = 0$$

Hence if $x(p) = 0$ for some $p \in M$,

then $\omega(x(p)) := \omega(x)(p) = 0$.

Hence ω_p defined as above makes sense.

→ Let $\mathcal{D}'(p) := \{ \omega_p \mid \omega \in \mathcal{D}'(M) \}$.

→ $\mathcal{D}'(p) = T_p^* M$. Clear: $\mathcal{D}'(p) \subseteq T_p^* M$.

Conversely, for $\varphi \in T_p^* M$, set $\varphi_i = \varphi\left(\frac{\partial}{\partial \alpha_i}\right)_p$.

$\Psi_U := \sum_i \varphi_i d\alpha_i$ on $(U, \alpha) \ni p$. Then

→ $\exists \psi \in \mathcal{D}'(M)$ s.t. $\Psi_U = \psi$ on an open

$V \subseteq U$, $p \in V$.

For this choose $V \ni p$, $\bar{V} \subseteq U$ compact; f smooth on M s.t. f has compact support $\subseteq U$, $f = 1$ on V . Define ψ by $\psi(x) := f \Psi_U(x)$.

21-5

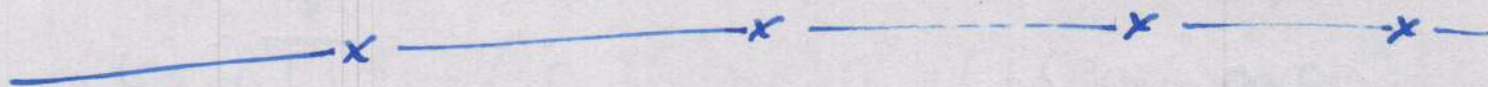
on $U \subset \psi(x)=0$ on $M \setminus U$. etc.

$$\rightarrow \psi_p = \varphi.$$

Hence $\mathcal{D}^1(M)$ consists of elements ω which can be thought of as a map $p \mapsto \omega_p \in T_p^*M$.

$$, \omega: M \rightarrow \bigcup_p T_p^*M, p \mapsto \omega_p \in T_p^*M.$$

We'll define higher diff forms next.



Lecture-22

In the last lecture we discussed the space of all differential 1-forms on M , as the set $D^1(M)$ consisting of elements ω , thought of as a map $p \mapsto \omega_p$, $\omega: M \rightarrow \bigcup_p T_p^* M$ ← Cotangent bundle of M , $\omega_p \in T_p^* M$.

Higher degree differential forms :- Recall the $C^\infty(M)$ -module $D_1(M) = \mathcal{X}(M)$. We define a differential form of degree r on M to be an r -linear alternating map of $C^\infty(M)$ -modules $\omega: D_1(M) \times \dots \times D_1(M) \rightarrow C^\infty(M)$. We denote the set of all deg r diff forms on M by $D^r(M)$.

Example :- Let $\omega \in D^1(M)$. Define a 2-form $d\omega: D_1(M) \times D_1(M) \rightarrow C^\infty(M)$ by $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

Then $d\omega \in D^2(M)$.

Exercise: We defined, for $f \in C^\infty(M)$ the 1-form $df \in D^1(M)$. What is $d(df)$?

22-2

This example is a particular case of

The d-operator :- We'll define $d: D^r(M) \rightarrow D^{r+1}(M)$

For this, we use (any) local expression for ω . Let (U, α) be a coordinate chart on M . Then ω has an expression

$$\omega = \sum_{i_1 \dots i_r} a_{i_1 \dots i_r} d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r} \quad \text{on } U.$$

($\omega_p \in \wedge^r T_p^* M$, $\omega_p(v_1, \dots, v_r) = \omega(x_1, \dots, x_r)(p)$, $x_i(p) = v_i$ on U etc.)

Define

$$d^L \omega := \sum da_{i_1 \dots i_r} \wedge d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r}$$

(here L stands for 'local').

We would like to extend the defⁿ of d^L to all of M . So we must verify, for another chart (V, β) on M with $U \cap V \neq \emptyset$, if

$$\omega = \sum b_{j_1 \dots j_r} d\beta_{j_1} \wedge \dots \wedge d\beta_{j_r} \quad \text{on } V, \text{ then}$$

$$\sum db_{j_1 \dots j_r} \wedge d\beta_{j_1} \wedge \dots \wedge d\beta_{j_r} = \sum da_{i_1 \dots i_r} \wedge d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r}$$

on $U \cap V$.

Global definition of d : We'll define $d: \bigoplus_{r=0}^m D^r M$

$\rightarrow \bigoplus D^r M$ and show it matches with d^L locally.

We have $D^0 M = C^\infty(M)$. Let $\omega \in D^r(M)$, $r \geq 1$ and $x_1, \dots, x_{r+1} \in D_1(M)$. Define

$$d\omega(x_1, \dots, x_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} x_i \omega(x_1, \dots, \hat{x}_i, \dots, x_{r+1}) \\ + \sum_{i < j} (-1)^{i+j} \omega([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{r+1})$$

$\rightarrow d\omega$ is additive clearly.

• $C^\infty(M)$ -linearity: $d\omega(x_1, \dots, f x_k, \dots, x_{r+1}) \stackrel{?}{=} f d\omega(x_1, x_2, \dots, x_k, \dots, x_{r+1})$.

• So $d\omega$ is $C^\infty(M)$ -linear

• $d\omega$ is alternating, so $d\omega \in D^{r+1}(M)$.

*₁: $\forall f \in C^\infty(M)$, $df(x) = X(f)$.

*₂: d is $\mathbb{R}$ -linear, $d(\alpha\omega + \beta\eta) = \alpha d\omega + \beta d\eta$

*₃: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ for $\omega \in D^r(M)$, $\eta \in D^s(M)$.

*₄: $d^2 = 0$, i.e. $d(d\omega) = 0 \neq \omega$.

Proposition :- Any $C^\infty(M)$ -linear operator

$d': D^r(M) \rightarrow D^{r+1}(M)$ satisfying the properties

*₁ - *₄ must coincide with d^L locally.

Proof:- We have

$$\begin{aligned} d'(d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r}) &= d'(d\alpha_{i_1}) \wedge d\alpha_{i_2} \wedge \dots \wedge d\alpha_{i_r} \\ &\quad - d\alpha_{i_1} \wedge d'(d\alpha_{i_2}) \wedge d\alpha_{i_3} \wedge \dots \wedge d\alpha_{i_r} \\ &\quad + \dots \end{aligned}$$

$$\text{We have } d'f = df \quad \forall f \in C^\infty(M) \quad (*_1)$$

$$\Rightarrow d'(d\alpha_{i_k}) = d'^2(\alpha_{i_k}) = 0 \quad \forall k \quad (*_4). \text{ Hence}$$

$$\begin{aligned} d'(f d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r}) &= df \wedge d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r} \\ &= d^L(f d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r}). \end{aligned}$$

Hence, to check the listed properties, it is necessary that d^L satisfies the properties on any chart (U, x) . We have, for $f \in C^\infty(M)$, $d^L(f) = df$ (by defⁿ of d^L), and, we have seen that for any $X \in \mathcal{X}(M) = \mathcal{D}_1(M)$, $df(X) = X(f)$. So $*_1$ is satisfied by d^L , $*_2$ is obvious. For $*_3$, using linearity, it suffices to check $*_3$ for $\omega = a d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r}$ & $\eta = b d\alpha_{j_1} \wedge \dots \wedge d\alpha_{j_s}$. Then $\omega \wedge \eta = a \cdot b d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_r} \wedge d\alpha_{j_1} \wedge \dots \wedge d\alpha_{j_s}$. □

$$\begin{aligned}
 \text{Hence } d^L(\omega \wedge \eta) &= d(ab) \wedge dx_{i_1} \wedge \dots \wedge dx_{j_s} \\
 &= (da \cdot b + a \cdot db) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_s} \\
 &= da \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r} \wedge b dx_{j_1} \wedge \dots \wedge dx_{j_s} \\
 &\quad + a dx_{i_1} \wedge \dots \wedge dx_{i_r} \wedge (-1)^r db \wedge dx_{j_1} \wedge \dots \wedge dx_{j_s} \\
 &= d\omega \wedge \eta + (-1)^r \omega \wedge d\eta. \quad \text{Hence } *_{\mathfrak{z}}
 \end{aligned}$$

holds for d^L .

*₄: Exercise: Using $\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}$ and

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad \text{it follows that}$$

$$d_L^2(f) = 0 \quad \forall f \in C^\infty(M). \quad \square$$

Since ω in general is a sum of (wedge) products of r 1-forms of type df , using $*_{\mathfrak{z}}$, every term of $d_L^2(\omega)$ has a factor of the form $d^L^2(f) = 0$, hence it follows that $d_L^2(\omega) = 0$. (check details) $\square$

Theorem: $d: \mathcal{D}^r(M) \rightarrow \mathcal{D}^{r+1}(M)$ defined by

$$\begin{aligned}
 d\omega(x_1, \dots, x_{r+1}) &= \sum_{i=1}^{r+1} (-1)^{i+1} \omega(x_1, \dots, \hat{x}_i, \dots, x_{r+1}) \\
 &\quad + \sum_{i < j} (-1)^{i+j} \omega([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{r+1})
 \end{aligned}$$

²²⁻⁶
 d matches with d^L on local charts.

Sketch :- Using linearity of d & d^L , it reduces to check, for $\omega = a dx_1 \wedge \dots \wedge dx_r$,

$$d\omega\left(\underbrace{\frac{\partial}{\partial x_{i_1}}}_{x_{i_1}}, \dots, \underbrace{\frac{\partial}{\partial x_{i_r}}}_{x_{i_r}}\right) = d^L \omega\left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_r}}\right)$$

This follows by explicit computations. $\square$

Pull back of a differential form :- Let-

$\varphi: M \rightarrow N$ be a smooth map, δ a diff. form on N . We define the pull back $\varphi^*(\delta)$ to be the differential form on M , having same degree as δ (say s), by

$$\begin{aligned} \varphi^*(\delta)(x_1, \dots, x_s)(p) \\ = \delta_{\varphi(p)}(D\varphi(p)(x_1(p)), \dots, D\varphi(p)(x_s(p))). \end{aligned}$$

Then $\varphi^*(\delta) \in \mathcal{D}^s(M)$. So $\varphi^*: \mathcal{D}^s(N) \rightarrow \mathcal{D}^s(M)$

s.t.o. We have

Propⁿ :- $\varphi: M \rightarrow N$ smooth $\omega \in \mathcal{D}(N)$. Then
 $d(\varphi^*\omega) = \varphi^*(d\omega) \rightsquigarrow$

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i.e. exterior derivative operator d commutes with pull-backs.

Pf: We may check this locally. So let

$$\omega = a dy_1 \wedge \dots \wedge dy_r. \text{ Then } d\omega = da \wedge dy_1 \wedge \dots \wedge dy_r$$

and $\varphi^*(d\omega) = \varphi^*(da \wedge dy_1 \wedge \dots \wedge dy_r)$

$$= \varphi^* da \wedge \varphi^* dy_1 \wedge \dots \wedge \varphi^* dy_r$$

Writing $\varphi_i = y_i \circ \varphi$

$$= d(a \circ \varphi) \wedge d\varphi_1 \wedge \dots \wedge d\varphi_r.$$

Also $\varphi^* \omega = \varphi^*(a dy_1 \wedge \dots \wedge dy_r)$

$$= \varphi^*(a) \varphi^* dy_1 \wedge \dots \wedge \varphi^* dy_r$$
$$= a \circ \varphi \cdot d\varphi_1 \wedge \dots \wedge d\varphi_r. \text{ Thus}$$

$$d(\varphi^* \omega) = d(a \circ \varphi) \wedge d\varphi_1 \wedge \dots \wedge d\varphi_r.$$

Hence φ^* and d commute.

□



Lecture-23

So far, we have defined differential forms of degree r , $r \geq 0$; their space is denoted by $D^r(M)$. We have defined a d -operator: $D^r(M) \xrightarrow{d_r} D^{r+1}(M)$ having the property $d^2 = 0$, i.e. $d_{r+1} \circ d_r = 0$, $r \geq 0$. Note also that d is an $\mathbb{R}$ -linear map.

Cochain complexes & their cohomology groups:

Let $\mathcal{C}: 0 \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^i \xrightarrow{d_{i-1}} C^{i+1} \xrightarrow{d_i} C^{i+2} \rightarrow \dots$

be a sequence of abelian groups and homomorphisms with $d_i \circ d_{i-1} = 0 \quad \forall i$, i.e.

$\text{Image } d_{i-1} \subseteq \text{Ker } d_i$. The i th cohomology group of $\mathcal{C}$ is defined by $H^i(\mathcal{C}) = \frac{\text{Ker } d_i}{\text{Im } (d_{i-1})}$.

Elements of $\text{Ker } d_i$ are called i -cocycles and elements of $\text{Im } (d_{i-1})$ are called i -coboundaries.

These cohomology groups measure the failure of $\mathcal{C}$ to be exact at C^i ; $H^i(\mathcal{C}) = 0 \Leftrightarrow \mathcal{C}$ is exact at C^i .

23-2

There is an analogous 'dual' notion of a chain complex: a sequence

$\mathcal{C}: \dots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \rightarrow \dots \xrightarrow{d_1} C_0 \rightarrow 0$ of abelian groups and homomorphisms, with $d_i \circ d_{i+1} = 0$ & decreasing degrees. One defines the i^{th} homology group $H_i(\mathcal{C}) = \frac{\text{Ker}(d_i)}{\text{Im}(d_{i+1})}$. This group measures the failure of $\mathcal{C}$ from being exact (i.e. $\text{Ker } d_i = \text{Im } d_{i+1}$) at C_i .

de-Rham cohomology groups: We use the co-chain complex

$\dots \rightarrow \mathcal{D}^i(M) \xrightarrow{d_i} \mathcal{D}^{i+1}(M) \rightarrow \dots$ of the space of differential forms. A diff form ω of degree k is called closed if $d\omega = 0$ and call ω exact if $\omega = d\eta$ for some $\eta \in \mathcal{D}^{k-1}(M)$. Since $d_i \circ d_{i-1} = 0$, it follows that exact forms are closed. Let

$$H_{dR}^k(M) = \text{Ker } d_k / \text{Im } d_{k-1}. \text{ This is}$$

the de-Rham cohomology of M in degree k .

23-3

We may sketch a proof of

Theorem :- (i) $H_{dR}^k(M)$ are finite dim'l $\mathbb{R}$ -vector spaces.

(ii) If $\beta_k = \dim_{\mathbb{R}} H_{dR}^k(M)$, then

$$\sum_k (-1)^k \beta_k = \chi(M), \text{ the Euler-}$$

Characteristic of M ; the number (integer) β_k is called the k^{th} Betti no. of M . $\boxtimes$

Orientation :- Recall, while handling 'limits of integration' we needed orientation. To do this for level surfaces $S = f^{-1}(c)$, we define S to be orientable if the map $S \rightarrow \mathbb{R}^{n+1}$ $p \mapsto N(p) := \nabla f(p)$ is smooth; where we assume $\nabla f(p) \neq 0 \forall p \in S$. In other words, the normal varies smoothly as we move on S .

Example : $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, $c=0$, $\nabla f(p) \neq 0 \forall p \in S = f^{-1}(0)$.

Then, for $p \in S$, $\nabla f(p) \perp T_p(S)$. Now $T_p(S)$ is

2 dim'l, say with basis $\{v_1, v_2\}$. Thus

$$\nabla f(p) \in \langle v_1 \wedge v_2 \rangle \quad \left| \begin{array}{l} \text{Exercize} \\ \leftarrow \end{array} \right| . \quad \curvearrowright$$

~~Moscow 2~~ $p \mapsto \nabla f(p)$ is smooth.

• Orientable manifold: A smooth manifold M is defined to be orientable if $\exists$ a smooth differential $m = \dim M$ -form ω on M such that $\omega_p \neq 0 \quad \forall p \in M$.

We call M oriented if M is orientable and a top degree nowhere vanishing diff. form ω is fixed, call ω the orientation class of M .

→ A basis $\{v_i, 1 \leq i \leq m\}$ of $T_p M$ is positively oriented if $\omega_p(v_1, \dots, v_m) > 0$.

• Orientability and coordinate systems: Let (M, ω) be an oriented manifold, $p \in M$ and let (U, α) be a coordinate chart around p ($\alpha(p) = 0$). For any $q \in U$, ω_q defines an orientation on $T_q M$ and we can define positively or negatively oriented coordinate systems.

Call (U, α) positively oriented if $\omega_q \left(\frac{\partial}{\partial x_1} \Big|_q, \dots, \frac{\partial}{\partial x_m} \Big|_q \right) > 0 \quad \forall q \in U$.

→ If $(x_1, \dots, x_m)$ is positively oriented then $(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ is negatively oriented for any odd $\sigma \in S_m$.

Exercise :- If U is connected, then

$q \mapsto \omega_q \left(\frac{\partial}{\partial x_1} \Big|_q, \dots, \frac{\partial}{\partial x_m} \Big|_q \right)$ has the same sign on U and we can write

$$\omega = f_U dx_1 \wedge \dots \wedge dx_m \text{ on } U \text{ with}$$

$$f_U(q) > 0 \forall q \text{ or } f_U(q) < 0 \forall q \in U.$$

Remark :- If (U, α) , (V, β) are two +vely (resp. -vely) oriented charts on M , then

$$\det \left(\frac{\partial \beta_i}{\partial \alpha_j} \right) > 0 \text{ on } U \cap V.$$

→ We call two charts (U, α) , (V, β) consistently oriented if either $U \cap V = \emptyset$ or $U \cap V \neq \emptyset$ and $\det \left(\frac{\partial \beta_i}{\partial \alpha_j} \right) > 0$ on $U \cap V$.

→ We call an atlas $\mathcal{A}$ an oriented atlas if all its charts are consistently oriented.

Thm : M is orientable $\Leftrightarrow M$ admits an oriented atlas.



Lecture-24

Integration :- We'll discuss now a theory of integration for differential forms on manifolds.

Let $U, V \subseteq \mathbb{R}^n$ be open, $\varphi: U \xrightarrow{\sim} V$ a C^1 -diffeomorphism. For any $f: V \rightarrow \mathbb{R}$ continuous such that $\text{Supp}(f) := \overline{\{x \in V \mid f(x) \neq 0\}}$ is compact, we have

$$\int_V f(y) dy = \int_U (f \circ \varphi) |\det D\varphi(x)| dx$$

More generally: for $U \subseteq \mathbb{R}^n$ open, $\varphi: U \rightarrow \mathbb{R}^n$ a 1-1 C^1 map with $\det(D\varphi(x)) \neq 0 \forall x \in U$, if $f: \varphi(U) \rightarrow \mathbb{R}$ is integrable, then

$$\bullet \int_{\varphi(U)} f(y) dy = \int_U (f \circ \varphi) |\det(D\varphi(x))| dx.$$

This is called the change of variable formula.

Exercise: Assume U , hence $V := \varphi(U)$ are

24-2
 Connect $\rightarrow$. (~~φ is assumed to be~~) Then φ is
orientation preserving or orientation reversing
 i.e. $\det(D\varphi(x)) > 0 \quad \forall x$ or $\det(D\varphi(x)) < 0$
 $\forall x \in U$.

Hence, in the case U is connect- ω ,

$$\int_{V=\varphi(U)} f(y) dy = \pm \int_U (f \circ \varphi) \det(D\varphi(x)) dx.$$

Next, suppose f is smooth and consider
 the differential form $\omega := f(y) dy_1 \wedge \dots \wedge dy_n$
 on V , y_i the usual coordinates on $\mathbb{R}^n|_V$.

The pull back $\varphi^*(\omega)$ is then

$$\varphi^*(\omega) = (f \circ \varphi) \varphi^*(dy_1 \wedge \dots \wedge dy_n)$$

$$= (f \circ \varphi) d(x_1 \wedge \dots \wedge x_n)$$

Now $\varphi^*: \bigwedge^n (T_{\varphi(p)} V)^* \rightarrow \bigwedge^n (T_p U)^*$ is

an isomorphism. Hence the basis elements

$$\varphi^*(dy_1 \wedge \dots \wedge dy_n) \text{ and } dx_1 \wedge \dots \wedge dx_n$$

are related as

$$\varphi^*(dy_1 \wedge \dots \wedge dy_n) = \det D\varphi(x) \cdot dx_1 \wedge \dots \wedge dx_n$$

So $\varphi^*(\omega) = (f \circ \varphi) \det(D\varphi(x)) dx_1 \wedge \dots \wedge dx_n$

We can thus interpret / rewrite the Change of variable formula as (6)

$$\int_{V=\varphi(U)} \omega = \pm \int_U \varphi^*(\omega)$$

On M , we can define the integral of a smooth differential form ω of degree $m = \dim M$, so long as support (ω)

$$:= \{p \in M \mid \omega_p \neq 0\}, \text{ is compact,}$$

$\text{Supp } (\omega) \subseteq (U, \alpha)$, the local chart being (U, φ^{-1}) ; we can write

$$\omega = f dx_1 \wedge \dots \wedge dx_m \text{ and define}$$

$$\int_M \omega := \int_{\varphi(U)} f(\varphi(x)) dx_1 \wedge \dots \wedge dx_m; \quad \text{--- (1)}$$

The RHS = Riemann int = Lebesgue int of $f \circ \varphi$.
Since $\text{Supp}(f \circ \varphi) \subseteq \varphi(U)$, this is finite.

This is well defined : If (V, ψ^{-1}) is another Chart with $\text{Supp}(\omega) \subseteq V$, then

$$\int_M \omega = \int_{\psi^{-1}(V)} f(\psi(p)) dy_1 \cdots dy_m, \quad y_i \text{ coords on } V. \quad \text{--- (2)}$$

If (V, ψ^{-1}) & (U, φ^{-1}) are *consistently oriented, then the RHS of (1) & (2)

are equal, using (0). ~~*~~

(i.e. $\det(\frac{\partial x_i}{\partial y_j}) > 0$).

In this case $\varphi \circ \psi^{-1}: \varphi^{-1}(U \cap V) \rightarrow \psi^{-1}(U \cap V)$ is orientⁿ preserving &

$$\int_{\psi^{-1}(U \cap V)} \omega = \int_{\varphi^{-1}(U \cap V)} \varphi^* \omega \quad \text{this is}$$

→ Therefore to define integral of a top degree form having compact support, we need M orientable.

$$U, V \subseteq \mathbb{R}^n \text{ open}, \varphi: U \xrightarrow{\sim} V \text{ diffeo}, \omega \in \mathcal{D}^n(V), \text{ then } \int_V \omega = \pm \int_U \varphi^* \omega$$

• We assume now on, to develop the notion of $\int$ that M is orientable.

$$\text{Cor: } U, V \subseteq \mathbb{R}^n \text{ open}, \varphi: U \xrightarrow{\sim} V \text{ orient preserving, then } \int_V \omega = \int_U \varphi^* \omega$$

→ We extend the above defⁿ to forms whose support are not nec. contained in a local chart, using partitions of unity.

Defⁿ: A family of C^r -maps $\{\varphi_\alpha\}, \varphi_\alpha: M \rightarrow \mathbb{R}$
 $\alpha \in \Lambda$

is a C^r -partition of unity subordinate to a covering $\{U_\beta\}_\beta$ of M if

(i) $\text{Supp}\{\varphi_\alpha\} \subseteq U_\beta$. (ii) $\{\text{Supp}\{\varphi_\alpha\}\}_\alpha$ is a locally finite cover of M (i.e. every $p \in M$ has an open nbd that intersects only finitely many of $\text{Supp}\{\varphi_\alpha\}$'s.)

(iii)
$$\sum_\alpha \varphi_\alpha(x) = 1 \quad \forall x \in M.$$

Thm: For any open covering of a smooth manifold, $\exists$ a partition of unity subordinate to it. $\square$

For M oriented, let $\mathcal{A}$ be a tri'ly oriented atlas, ω a top degree form on M . Let $\{f_i\}$ be a smooth partition of unity subordinate to $\mathcal{A}$. Fix tri'ly oriented coords on the charts in $\mathcal{A}$. We set

$$\int_M \omega := \sum_i \int_{U_i} f_i \omega.$$

→ Each of $\int_{U_i} f_i \omega$ is finite and the sum is over a finite number of indices.

Lecture-25

Stokes theorem :- Let M be a manifold, $\Omega \subseteq M$ a connected open subset, $\partial\Omega$ its topological boundary in M . Roughly, Stokes theorem states that

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega \quad \text{for } \omega \in \mathcal{D}^{m-1}(M).$$

We'll shortly give a more strict & precise statement.

We'll call $\Omega \subseteq M$, Ω connected & open as a domain in M . Recall that the topological boundary of Ω is given by the set $\partial\Omega \subseteq M$ consisting of those points in M whose any open neighbourhoods intersect Ω as well as $M \setminus \Omega$.

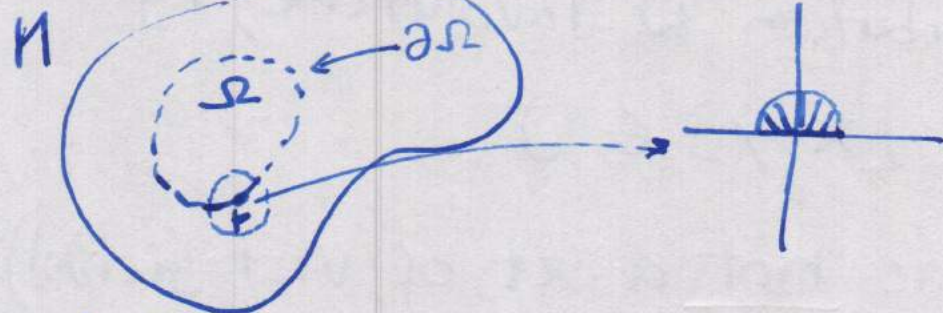
We say Ω has smooth boundary if for every $p \in \partial\Omega$, $\exists$ a local chart (U, α) around p , with $\alpha_i(p) = 0$, $1 \leq i \leq m = \dim M$, such that

$$U \cap \overline{\Omega} = \{z \in U \mid \alpha_m(z) \geq 0\},$$

i.e. α_m maps $U \cap \overline{\Omega}$ into the upper half space

$$\mathbb{R}_+^m := \{y \in \mathbb{R}^m \mid y_m \geq 0\}.$$

25-2



We have

Theorem:- Let Ω be a domain in M , assume Ω has smooth boundary in M . Then $\partial\Omega$ is an $(m-1)$ -dim'l closed submanifold of M . Moreover, if M is orientable, $\partial\Omega$ is orientable. $\square$

Starting with an oriented manifold M , we wish to fix an orientation on $\partial\Omega$, where Ω is a domain $\subseteq M$. First let $M = \mathbb{R}^{n+1}$

$$\& \quad \Omega = \mathbb{R}_{\geq 0}^{n+1} = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} \geq 0\}. \text{ Let}$$

$\{e_i\}$ be the standard basis of $\mathbb{R}^{n+1}$ and

$\{u_i\}$ its dual basis. Then

$$\overline{\Omega} = \{x \in \mathbb{R}^{n+1} \mid u_{n+1}(x) \geq 0\},$$

$$\partial\Omega = \{x \in \mathbb{R}^{n+1} \mid u_{n+1}(x) = 0\} \simeq \mathbb{R}^n.$$

On $\mathbb{R}^{n+1}$ the orientation is given by the $n+1$ -volume form $dx_1 \wedge \cdots \wedge dx_{n+1}$.

²⁵⁻³ We identify $T_x \mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ so that on each $T_x \mathbb{R}^{n+1}$ we have the form $u_1 \wedge \dots \wedge u_{n+1}$ corresponding to $dx_1 \wedge \dots \wedge dx_{n+1} |_x$.

The 'outward normal' to $\partial\Omega$ is given by the vector field $p \mapsto -\frac{\partial}{\partial x_{n+1}} |_p$; with the usual identifications, this is $-e_{n+1}$.

On $T_p(\partial\Omega)$ we fix an orientation by requiring that $\{v_1, \dots, v_n\}$, a basis of $T_p(\partial\Omega)$, is positively oriented if

$$(u_1 \wedge \dots \wedge u_{n+1})(-e_{n+1}, v_1, \dots, v_n) > 0.$$

(e.g. for $v_i := e_i$, we get

$$= \det \begin{pmatrix} -u_1(e_{n+1}) & u_1(e_1) & \dots & u_1(e_n) \\ -u_2(e_{n+1}) & u_2(e_1) & \dots & u_2(e_n) \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n+1}(e_{n+1}) & u_{n+1}(e_1) & \dots & u_{n+1}(e_n) \end{pmatrix}$$

$$= (-1)^{n+1}; \quad > 0 \Leftrightarrow n \text{ is odd.}$$

Thus a positive orientation for $\partial\Omega$ is given by the form $dx_1 \wedge \dots \wedge dx_n$ if n odd & by $-dx_1 \wedge \dots \wedge dx_n$, if n is even.

Now, for $\Omega \subseteq M$ a domain with smooth boundary, for $p \in \partial\Omega$, $\exists$ a +ve'ly orient- ω local chart $(U, \alpha) \ni p$ in M s.t.

$$U \cap \partial\Omega = \{z \in U \mid \alpha_m(z) = \alpha_m(p)\}.$$

We require, on this chart, that the orientation is given by

Exc: The ordered basis $\{v_1, \dots, v_{m-1}\}$ of $T_p(\partial\Omega)$ is +ve'ly orient- ω if $\{N_p, v_1, \dots, v_{m-1}\}$ is an +ve'ly orient- ω , $N_p = -e_m$. (?)

$$\begin{cases} dy_1 \wedge \dots \wedge dy_{m-1} & \text{if } m-1 \text{ is even} \\ & \text{(so } m \text{ odd)} \\ -dy_1 \wedge \dots \wedge dy_{m-1} & \text{if } m-1 \text{ is odd} \\ & \text{(so } m \text{ even)} \end{cases}$$

If we fix this orientation on $\partial\Omega$, we say that $\partial\Omega$ has a compatible orientation.

We now can state

Stokes' theorem :- Let M be an orient- ω manifold, $\Omega \subseteq M$ a domain with smooth boundary, $\bar{\Omega}$ compact. Let $\omega \in \mathcal{D}^{m-1}(M)$,

$i: \partial\Omega \rightarrow M$ an embedding. Let $\partial\Omega$ have the compatible boundary orientation.

Then
$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^*(\omega).$$



25-5
Applications: Stokes theorem generalizes
the fundamental theorem of calculus:

$$\int_a^b f'(x) dx = f(b) - f(a). \text{ Here } \Omega = [a, b],$$

a 1-manifold with boundary $\partial\Omega = \{a, b\}$.

Green's theorem: $D \subseteq \mathbb{R}^2$ a domain with
smooth boundary, P, Q smooth functions on
 D . Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} \underbrace{(P dx + Q dy)}_{\omega}$$

Pf: We have $d(P dx + Q dy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$

Hence, by defⁿ of $\int$ of forms, " $d\omega$ "

$$\begin{aligned} \int_{\partial D} \omega &= \int_D d\omega = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \end{aligned}$$

25-6

Cor: Let M be a compact manifold without boundary ($\partial M = \emptyset$). Then for any $(m-1)$ -form ω , $\int_M d\omega = 0$.

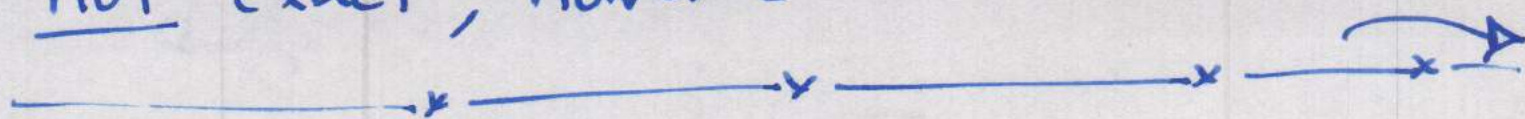
Cor:- Let M be a compact manifold with ^{smooth} boundary and ω a closed $(m-1)$ -form on M (i.e. $d\omega = 0$). Then $\int_{\partial M} \omega = 0$.

Pf: $\int_{\partial M} \omega = \int_M d\omega = 0$.

Exercise: Let $\omega = \frac{-y dx + x dy}{x^2 + y^2}$ on S^1 .

Then $\int_{S^1} \omega = 2\pi$.

By the above, we conclude that ω is not exact, however ω is closed (exc).



25-7:

Lemma: Let M be compact orient'd. Then $\exists$ a diff. form ω of deg m s.t. every local representative form is positive everywhere on M .

Pf (Sketch): $\exists$ an orient'd atlas with finitely many charts $\{V_i\}$. Let $\{\varphi_i\}$ be a partⁿ of unity subordinate to this. On each V_i , define $\omega_i = dx_1 \wedge \dots \wedge dx_m$. Since the atlas is orient'd, these remain +ve on overlaps of the V_i 's.

Define $\omega = \sum \varphi_i \omega_i$. This is > 0 Δ global. $\square$

Theorem: There is no smooth retraction from a compact orient'd manifold to its boundary.

Pf: (Recall: $r: M \rightarrow \partial M$ is a retraction if $r|_{\partial M} = \text{id}_{\partial M}$).

Let ω be as in the Lemma. Since ∂M is $(m-1)$ dim'l, $d\omega = 0$ on ∂M . Suppose $r: M \rightarrow \partial M$ is a smooth retraction. We have $d(r^*\omega)$

$= r^*(d\omega) = 0$. Since $r|_{\partial M} = \text{id}_{\partial M}$, we

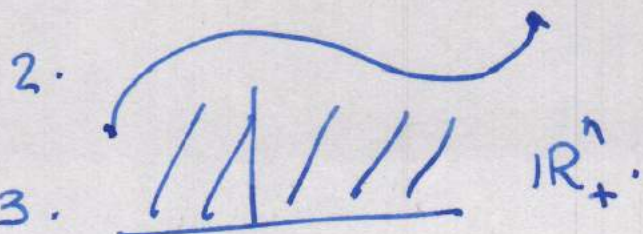
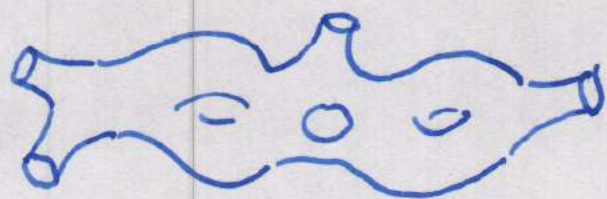
have $r^*(\omega)|_{\partial M} = \omega$. Hence $\int_M \omega = \int_M r^*\omega$
 $= \int_M d(r^*\omega) = \int_M r^*(d\omega) = 0 \rightarrow \text{to } \int_{\partial M} \omega > 0.$

Let us clarify a couple of ideas we discussed in the last class:

$$\mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\} \leftarrow \text{upper half space in } \mathbb{R}^n.$$

- A 'smooth manifold with boundary' is a T_2 -Space M , having a countable basis of open sets & a 'differentiable structure' $\mathcal{A}$ in the following sense: $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is a family of open subsets U_α of M , with homeomorphisms φ_α onto an open subset of $\mathbb{R}_+^n$ (which has the subspace topology from $\mathbb{R}^n$) such that
 - i) $\bigcup_\alpha U_\alpha = M$ (ii) for $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathcal{A}$, with $U_\alpha \cap U_\beta \neq \emptyset$, the maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ & $\varphi_\alpha \circ \varphi_\beta^{-1}$ are 'diffeomorphisms' of $\varphi_\alpha(U_\alpha \cap U_\beta)$ & $\varphi_\beta(U_\alpha \cap U_\beta)$, $\subseteq \mathbb{R}_+^n$.
 - (iii) $\mathcal{A}$ is maximal satisfying (i) & (ii).

Examples: 1. Tori with 'discs' removed



Remarks $\therefore \partial \mathbb{R}_+^n = \{x \in \mathbb{R}_+^n \mid x_n = 0\} \equiv \mathbb{R}^{n-1}$.

• If (U, φ) is a coordinate chart in M with boundary and if for $p \in U$, $\varphi(p) \in \partial \mathbb{R}_+^n$, then for any coordinate chart $(V, \psi) \ni p$, $\psi(p) \in \partial \mathbb{R}_+^n$ (Why?). We define the boundary of M by

$$\partial M = \{p \in M \mid \varphi(p) \in \partial \mathbb{R}_+^n \text{ for some chart } (U, \varphi) \ni p\}.$$

• Usual manifolds are regarded as manifolds with boundary with $\partial M = \emptyset$. In this case we also say M is a manifold without boundary. We mentioned last time that for M with $\partial M \neq \emptyset$, $\dim M = n$, ∂M carries a diff. structure determined by that on M and $\dim \partial M = n-1$; the inclusion $i: \partial M \rightarrow M$ is an embedding.

• When $\partial M = \emptyset$ & $\omega \in \mathcal{D}^{n-1}(M)$, we define

$$\int_{\partial M} \omega = 0.$$

We'll now prove Stokes Theorem $\leadsto$

Theorem :- M be an oriented manifold, $\Omega \subseteq M$ a domain with smooth boundary, $\bar{\Omega}$ compact. Let $m = \dim M$, $\omega \in \mathcal{D}^{m-1}(M)$, $i: \partial\Omega \rightarrow M$ an embedding and $\partial\Omega$ has the compatible orientation. Then

$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^*(\omega).$$

Proof :- We can choose a locally finite covering

$\{U_{\alpha}\}$ of M such that

i) U_{α} is a chart $\forall \alpha$.

ii) If $U_{\alpha} \cap \bar{\Omega} \neq \emptyset$ then either $U_{\alpha} \subseteq \Omega$

or $U_{\alpha} \cap \partial\Omega \neq \emptyset$. If $U_{\alpha} \cap \partial\Omega \neq \emptyset$ we

(may) assume $U_{\alpha} \subseteq \{p \mid |x_i^{\alpha}(p)| < \epsilon, 1 \leq i \leq m\}$;

(recall $U_{\alpha} \cap \bar{\Omega} = \{q \mid x_m^{\alpha}(q) \geq 0\}$), for any such U_{α} .

Let $\{f_{\alpha}\}$ be a partition of unity subordinate to $\{U_{\alpha}\}$. Since $\partial\Omega$ & $\bar{\Omega}$ are compact &

$\{U_{\alpha}\}$ locally finite, the set $\{\alpha \mid U_{\alpha} \cap \partial\Omega \neq \emptyset$
or $U_{\alpha} \cap \bar{\Omega} \neq \emptyset\}$

is a finite set.

Hence $\int_{\Omega} d\omega = \sum_{\alpha} \int_{\Omega} d(f_{\alpha}\omega)$, $\omega = \sum_{\alpha} f_{\alpha}\omega$

and $\int_{\partial\Omega} i^*(\omega) = \sum_{\alpha} \int_{\partial\Omega} i^*(f_{\alpha}\omega)$.

→ Note that $\text{Supp}(f_{\alpha}\omega) \subseteq U_{\alpha}$.

→ This reduces to proving: if M is a compact manifold (oriented) and $\partial\Omega$ has the compatible boundary orientation, then

$$\int_{\partial\Omega} i^*(\omega) = \int_{\Omega} d\omega, \text{ where } \text{Support}(\omega) \subseteq U_{\alpha} \text{ for some } U_{\alpha} =: U.$$

→ If $U \cap \bar{\Omega} = \emptyset$ then ω and $d\omega$ both are zero on $\bar{\Omega}$ (why?). Therefore both the integrals above are zero, so equal.

→ Assume therefore $U \cap \bar{\Omega} \neq \emptyset$. ~~If~~

First let us consider the case $U \subseteq \Omega$. Then

$$U \cap \partial\Omega = \emptyset, \text{ hence } \int_{\partial\Omega} \omega = 0.$$

Since ω is an $(n-1)$ form on M with $\text{Supp}(\omega) \subseteq U$,

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we have

$$\omega = \sum_{k=1}^m (-1)^{k-1} f_k dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_m$$

$$\text{Hence } d\omega = \sum_{k=1}^m (-1)^{k-1} df_k \wedge dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_m$$

$$= \sum_{k=1}^m (-1)^{k-1} \left(\sum_{j=1}^m \frac{\partial f_k}{\partial x_j} dx_j \right) \wedge dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_m$$

$$= \sum_{k=1}^m \frac{\partial f_k}{\partial x_k} dx_1 \wedge \dots \wedge dx_m. \quad \text{Hence}$$

$$\int_U d\omega = \sum_k \int_{\mathbb{R}^m} \frac{\partial f_k}{\partial x_k} dx_1 \dots dx_m.$$

$$\text{Now } \int_{\mathbb{R}^m} \frac{\partial f_k}{\partial x_k} dx_1 \dots dx_m = \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_k}{\partial x_k} dx_k \right) dx_1 \dots \widehat{dx_k} \dots dx_m.$$

Note that $\int_{-\infty}^{\infty} \frac{\partial f_k}{\partial x_k} dx_k$ is a function of the

variables $x_1, \dots, \widehat{x_k}, \dots, x_m$; given by

$$(a_1, \dots, \widehat{a_k}, \dots, a_m) \mapsto \int_{-\infty}^{\infty} g'_k(t) dt, \text{ where}$$

$$g_k(t) = f_k(x_1, \dots, t, \dots, x_m).$$

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Now f_k has compact support, hence g_k vanishes outside any suff. large interval $(-b, b) \subseteq \mathbb{R}$.

By the fundamental thm of calculus,

$$\int_{-\infty}^{\infty} g'_k(t) dt = \int_{-b}^b g'_k(t) dt = \underbrace{g_k(b)}_0 - \underbrace{g_k(-b)}_0 = 0.$$

It therefore follows that $\int_U d\omega = 0 = \int_{\Omega} d\omega$.

→ Now consider the case $U \not\subseteq \Omega$. Then ~~even~~ $U \cap \partial\Omega \neq \emptyset$. Observe that for $k \neq m$ the integration limits for g'_k are symmetric

(since for $q \in U \cap \bar{\Omega}$, $x_m(q) \geq 0$), hence
 $\int_{-b}^b g'_k(t) dt = 0$ for b large, ^{by the above} ~~and the~~ argument.

When $k=m$, on $\partial\Omega$ we have $x_m=0$.

In this case we have

$$i^*(\omega) = (-1)^{m-1} f_m(x_1, \dots, x_{m-1}, 0) dx_1 \wedge \dots \wedge dx_{m-1} \text{ on } \partial\Omega.$$

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Hence

$$\int_U d\omega = \sum_{k=1}^m \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} \frac{\partial f_k}{\partial x_k} dx_1 \wedge \dots \wedge dx_m,$$

Where f_k vanishes if x_k is outside some large interval $(0, \epsilon)$. Although $f_k(x_1, \dots, x_{k-1}, \epsilon) = 0$, $f_k(x_1, \dots, x_{k-1}, 0) \neq 0$ (possibly).

Again, applying the fundamental thm of calculus we have

$$\begin{aligned} \int_U d\omega &= \int_{\mathbb{R}^{m-1}} \int_0^\epsilon \frac{\partial f_m}{\partial x_m} dx_1 \wedge \dots \wedge dx_{m-1} \wedge dx_m \\ &= \int_{\mathbb{R}^{m-1}} f_m(x_1, \dots, x_{m-1}, t) \Big|_0^\epsilon dx_1 \dots dx_{m-1} \\ &\quad - \int_{\mathbb{R}^{m-1}} f_m(x_1, \dots, x_{m-1}, 0) dx_1 \dots dx_{m-1} \\ &= \int_{\mathbb{R}^{m-1}} f_m(x_1, \dots, x_{m-1}, 0) dx_1 \dots dx_{m-1}. \end{aligned}$$

On the other hand, on $\partial\Omega$, $i^*(\omega) = f_m(x_1, \dots, x_{m-1}, 0)$.

→ Now $\partial\Omega$ has orientation given by the form $(-1)^m dx_1 \wedge \dots \wedge dx_{m-1}$. Therefore

$$\int_{\partial\Omega} i^*(\omega) = (-1)^m \int_{\mathbb{R}^{m-1}} (-1)^{m-1} f_m(x_1, \dots, x_{m-1}, 0) dx_1 \dots dx_{m-1}$$

$$= - \int_{\mathbb{R}^{m-1}} f_m(x_1, \dots, x_{m-1}, 0) dx_1 \dots dx_{m-1}. \quad \text{Hence we have}$$

$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^*(\omega).$$

