Function Spaces - B. Math. III

Assignment 1 — Odd Semester 2023-2024

Due date: August 29, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) For $n \in \mathbb{N}$, let $f_n \in C([0,1];\mathbb{R})$ be given by $f_n(x) = x^n$.
   (a) (5 points) Prove that the sequence $\{f_n\}$ converges pointwise but not uniformly.
   (b) (5 points) Let $g \in C([0,1];\mathbb{R})$ with $g(1) = 0$. Show that the sequence $\{x^n g(x)\}$ converges uniformly on $[0,1]$.
2. (10 points) Prove that $\sum_{n=1}^{\infty} x^n(1-x)$ converges pointwise but not uniformly on $[0,1]$, whereas $\sum_{n=1}^{\infty}(-1)^n x^n(1-x)$ converges uniformly on $[0,1]$. (This illustrates that uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$.)
3. (10 points) Let $\{a_n\}$ de a decreasing sequence of positive real numbers. Prove that the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$ if, and only if, $na_n \to 0$ as $n \to \infty$.
4. (10 points) Prove that the series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on the interval $[1+\varepsilon, \infty)$ for every $\varepsilon > 0$. Show that the equation $\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$ is valid for each $s > 1$ and obtain a similar formula for the kth derivative $\zeta^{(k)}(s)$.
5. (10 points) Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ such that $0 \le f_n(x) \le 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Prove that there is a function f and a subsequence $\{n_k\}$ such that
$$f(x) = \lim_{n\to\infty} f_{n_k}(x)$$
for all $x \in \mathbb{R}$.
6. (10 points) Prove that the unit ball of $(C[0,1], \|\cdot\|_{\infty})$ is not compact.

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