Function Spaces - B. Math. III
Assignment 2 — Odd Semester 2023-2024

Due date: September 7, 2023
Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

For $A \subseteq \mathbb{R}^n$, we denote the convex hull of A by $\text{conv}(A)$. In other words, $\text{conv}(A)$ is the smallest convex set of $\mathbb{R}^n$ containing A.

For a convex set $K \subseteq \mathbb{R}^n$, a point $x \in K$ is said to be an *extreme point* for K if it cannot be written as a non-trivial convex combination of two distinct elements of K, that is, $x = ty + (1-t)z$ for $t \in (0,1), y, z \in K$ implies that $x = y = z$.

1. (10 points) (a) (5 points) Show that $A \subseteq \mathbb{R}^n$ is convex, if and only if $\alpha A + \beta A = (\alpha + \beta)A$ holds, for all $\alpha, \beta \geq 0$.
   (b) (5 points) Which non-empty sets $A \subseteq \mathbb{R}$ nare characterized by $\alpha A + \beta B = (\alpha + \beta)A$, for all $\alpha, \beta \in \mathbb{R}$?

2. (10 points) A set $R := \{x + \alpha y : \alpha \geq 0\}, x, y \in \mathbb{R}^n, \|y\| = 1$ is called a ray (starting in x with direction y).
   (a) (5 points) Let $A \subseteq \mathbb{R}^n$ be convex, closed and unbounded. Show that A contains a ray.
   (b) (5 points) In the above question, is it necessary to assume that A is a closed set?

3. (10 points) Let $A \subseteq \mathbb{R}^n$ be a locally finite set (this means that $A \cap B(0,r)$ is a finite set, for all $r \geq 0$, where $B(r)$ denote the closed ball of radius r centred at the origin). For each $x \in A$, we define the **Voronoi cell**,
$$C(x, A) := \{z \in \mathbb{R}^n : \|z - x\|_2 \leq \|z - y\|_2 \ \forall y \in A\},$$
   consisting of all points $z \in \mathbb{R}^n$ which have x as their nearest point (or one of their nearest points) in A.
   (a) (5 points) Let A be the set of vertices of a regular hexagon. Provide a rough sketch of the Voronoi cell of one of its vertices.
   (b) (5 points) If $\text{conv}(A) = \mathbb{R}^n$, show that the Voronoi cells are bounded.

4. (10 points) Prove that a compact convex set in $\mathbb{R}^2$ is the convex hull of its extreme points.

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5. (10 points) Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a linear functional. Prove that there is a unique vector $x_\rho \in \mathbb{R}^n$ such that $\rho(y) = \langle y, x_\rho \rangle$ for all $y \in \mathbb{R}^n$.

6. (10 points) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **convex** if for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$, we have $f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y)$. Moreover, f is called **concave** if $-f$ is convex. If f is both convex and concave, then f is called **affine**; In other words, for an affine function f, we have $f\big(tx + (1-t)y\big) = tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$.

   Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous concave function and $g : \mathbb{R}^n \to \mathbb{R}$ continuous convex function satisfying $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$. Show that there exists an affine function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}^n$.

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