Function Spaces - B. Math. III

Assignment 3 — Odd Semester 2023-2024

Due date: September 14, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) Let $\{f_n\}$ and $\{g_n\}$ be increasing sequences of functions on $[0,1]$. Let $u_n = \max\{f_n, g_n\}$ and $v_n = \min\{f_n, g_n\}$.

   (a) (5 points) Prove that $\{u_n\}$ and $\{v_n\}$ are increasing on $[0,1]$.

   (b) (5 points) If $f_n \uparrow f$ *a.e.* on $[0,1]$ and $g_n \uparrow g$ *a.e.* on $[0,1]$, prove that $u_n \uparrow \max(f,g)$ and $v_n \uparrow \min(f,g)$ *a.e.* on $[0,1]$.

2. (10 points) Let $\{s_n\}$ be an increasing sequence of step functions which converges pointwise on $[0,\infty)$ to a limit function f. If $f(x) \geq 1$ almost everywhere on $[0,\infty)$, prove that the sequence $\int_0^n s_n\, dx$ diverges.

3. (5 points) Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{n}$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$, and $f(0) = 0$. Show that f is an upper function and that $-f$ is not an upper function.

4. (15 points) Let $\{r_1, \ldots, r_n, \ldots\}$ be an enumeration of the set of rational numbers in $[0,1]$ and let $I_n := [r_n - \frac{1}{4^n}.r_n + \frac{1}{4^n}] \cap [0,1]$. Let $f(x) = 1$ if $x \in I_n$ for some n, and let $f(x) = 0$ otherwise.

   (a) (5 points) Show that $f \in U([0,1])$ and $\int_0^1 f\, dx \leq \frac{2}{3}$.

   (b) (5 points) If a step function s satisfies $s \leq -f$ on $[0,1]$, show that $s(x) \leq -1$ almost everywhere on $[0,1]$ and hence $\int_0^1 s\, dx \leq -1$.

   (c) (5 points) Show that $-f \notin U([0,1])$.

5. (10 points) Let $\mathbb{Q}$ denote the set of rational numbers. On the interval $[0,1]$, define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Show that f is Lebesgue integrable, with

$$\int_0^1 f(x)\, dx = 0,$$

   but it is **not** Riemann-integrable.

6. (10 points) If $f_n(x) = e^{-nx} - 2e^{-2nx}$, show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\, dx \neq \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx.$$

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