Function Spaces - B. Math. III

Assignment 4 — Odd Semester 2023-2024

Due date: September 14, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

A complex-valued function $f : [0,1] \to \mathbb{C}$ is said to be Lebesgue-integrable if $\Re f$ and $\Im f$ are both in $L^1([0,1]; dx)$. Thus we may expand the scope of $L^1([0,1]; dx)$ to include complex-valued Lebesgue-integrable functions.

For an open interval $(a,b) \subseteq [0,1]$, we define $|(a,b)| = b - a$ as the *size* of the interval. Since every relatively open set E in $[0,1]$ can be written uniquely as a (countable) disjoint union of open intervals, we use the notation $|E|$ for the sums of the sizes of the open intervals, and call it the size of E. Note that the size of E is less than or equal to 1.

1. (20 points) Prove that each of the following exists as a Lebesgue integral.
   (a) (5 points) $\int_0^1 \frac{x \log x}{(1+x)^2}\, dx$,
   (b) (5 points) $\int_0^1 \frac{x^p - 1}{\log x}\, dx \quad (p > -1)$,
   (c) (5 points) $\int_0^1 \log x \cdot \log(1+x)\, dx$,
   (d) (5 points) $\int_0^1 \frac{\log(1-x)}{\sqrt{1-x}}\, dx$.
2. (10 points) Assume that f is continuous on $[0,1]$, $f(0) = 0$, $f'(0)$ exists. Prove that the Lebesgue integral
$$\int_0^1 f(x) x^{-\frac{3}{2}}\, dx$$
exists.
3. (10 points) Let $f \in L^1([0,1]; dx))$. Show that for each $\varepsilon > 0$, there exist $\delta > 0$ (depending on ε) such that for any relatively open subset E of $[0,1]$ with $|E| < \delta$, we have
$$\Big| \int_E f\, dx \Big| := \Big| \int_0^1 \chi_E f\, dx \Big| < \varepsilon.$$
(In other words, the integral of a function in $L^1([0,1]; dx)$ is uniformly small on small open sets.)

Soumyashant Nayak: soumyashant@isibang.ac.in

4. (10 points) Let φ be a differentiable function on $\mathbb{R}$ with bounded derivative. If $f \in L^1([0,1]; dx)$, show that the function $\Psi : [0,1] \to \mathbb{R}$ defined by

$$\Psi(t) = \int_0^1 \varphi(tx) f(x) \, dx,$$

is differentiable, and

$$\Psi'(t) = \int_0^1 \varphi'(tx) x f(x) \, dx.$$

(Hint: Use Dominated Convergence Theorem.)

5. (10 points) (a) (5 points) Let $\chi_n : [0,1] \to \mathbb{C}$ be the function $\chi_n(x) = e^{2\pi i n x}$ and $f : [0,1] \to \mathbb{C}$ be a function. Prove that if $f\chi_k \in L^1([0,1]; dx)$ for some $k \in \mathbb{Z}$, then $f\chi_n \in L^1([0,1]; dx)$ for every $n \in \mathbb{Z}$.

   (b) (5 points) Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{n^{\frac{3}{2}} x}{1 + n^2 x^2} \, dx.$$

   (Hint: Use Dominated Convergence Theorem)

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