Function Spaces - B. Math. III

Assignment 5 — Odd Semester 2023-2024

Due date: October 3, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) (a) (5 points) Show that $\log \frac{1}{1-x} \in L^1([0,1]; dx)$ and with justification, compute the following integral:

$$\int_0^1 \log \frac{1}{1-x}\, dx.$$

   (b) (5 points) For $p > 0$, show that $\frac{x^{p-1}}{1-x} \log \frac{1}{x} \in L^1([0,1]; dx)$ and

$$\int_0^1 \frac{x^{p-1}}{1-x} \log \frac{1}{x}\, dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2}.$$

2. (15 points) Let $f : [0,1] \to \mathbb{R}$ be a function and $g : [0,1] \to \mathbb{R}$ by $g(x) = e^{f(x)}$.

   (a) (5 points) Show that if f is measurable, then so is g.

   (b) (5 points) If f is Lebesgue-integrable, is then g necessarily Lebesgue integrable? Prove or provide counterexample with justification.

   (c) (5 points) Give an example of an essentially unbounded function f which is continuous on $(0,1]$ such that f^n is Lebesgue-integrable for all positive integers n. (A function f is *essentially unbounded* if for every $M > 0$ the set $\{x \in [0,1] : |f(x)| > M\}$ is not negligible, that is, not of measure zero.)

3. (5 points) (Fundamental Theorem of Calculus) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function- with one-sided derivatives at the end-points 0 and 1. If the derivative f' is uniformly bounded on $[0,1]$, then show that f' is Lebesgue-integrable and that

$$\int_0^1 f'\, dx = f(1) - f(0).$$

Soumyashant Nayak: soumyashant@isibang.ac.in

4. (10 points) Let $f, g : [0,1] \to \mathbb{R}$ be two Lebesgue-integrable functions satisfying

$$\int_0^t f(x)\, dx \le \int_0^t g(x)\, dx,$$

for all $t \in [0,1]$. If $\varphi : [0,1] \to \mathbb{R}$ is a non-negative decreasing function, the show that the functions φf and φg are Lebesgue-integrable over $[0,1]$ and that they satisfy

$$\int_0^t \varphi(x) f(x)\, dx \le \int_0^t \varphi(x) g(x)\, dx$$

for all $t \in [0,1]$.

5. (10 points) For $t \ge 0$, let

$$A(t) := \Big(\int_0^t e^{-x^2}\, dx\Big)^2, B(t) := \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\, dx.$$

   (a) (5 points) Prove that $A(t) + B(t) = \frac{\pi}{4}$ for all $t \ge 0$. (Hint: What is $A'(t) + B'(t)$?)

   (b) (5 points) Prove that $e^{-x^2} \in L^1(\mathbb{R}_{\ge 0}; dx)$ and $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

   (N.B.: Carefully justify each step, such as existence of integral, interchange of limits and integrals, etc.)

6. (10 points) Show that for each $t \ge 0$, the integral $\int_0^\infty \frac{\sin xt}{x(x^2+1)}\, dx$ exists both as an improper Riemann integral and as a Lebesgue integral, and that

$$\int_0^\infty \frac{\sin xt}{x(x^2+1)}\, dx = \frac{\pi}{2}(1 - e^{-t}).$$

Soumyashant Nayak: soumyashant@isibang.ac.in