Function Spaces - B. Math. III

Assignment 6 — 1st Semester 2023-2024

Due date: October 23, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

For $f \in L^1[-\pi, \pi]$, the Fourier coefficients of f are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos nx\ dx, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx.$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the *Fourier series* of f, and we write

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

1. (30 points) Compute the Fourier series of the following functions appropriately defining them on $[-\pi, \pi]$ assuming they have period 2π.
   (a) (5 points) $f(x) = x$ on $[0, 2\pi]$
   (b) (5 points) $f(x) = x^2$ on $[0, 2\pi]$;
   (c) (5 points) $f(x) = x$ on $[-\pi, \pi]$;
   (d) (5 points) $f(x) = x^2$ on $[-\pi, \pi]$;
   (e) (5 points) $f(x) = \cos\frac{x}{2}$ on $[0, 2\pi]$;
   (f) (5 points) $f(x) = \sin\frac{x}{2}$ on $[0, 2\pi]$.
2. (10 points) Show that if $f, g \in L^1[-\pi, \pi]$ have the same Fourier series, then $f = g$ a.e. on $[-\pi, \pi]$.
3. (10 points) (a) (5 points) Provide a simple description of a continuous function on $[-\pi, \pi]$ which generates the Fourier series,

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin nx}{n^3}.$$

   (b) (5 points) Use Parseval's formula to conclude that $\zeta(6) = \frac{\pi^6}{945}$.
4. (10 points) (Smoothness and decay) Suppose that f is a 2π-periodic function that satisfied the Lipschitz condition of order α $(0 < \alpha \leq 1)$; that is $|f(x+h) - f(x)| \leq C|h|^\alpha$ for $C > 0$ independent of x. Show that if a_n, b_n are Fourier coefficients of f, then

$$a_n = O(n^{-\alpha}), b_n = O(n^{-\alpha}).$$

Statistics and Mathematics Unit, Indian Statistical Institute
Soumyashant Nayak: soumyashant@isibang.ac.in