Function Spaces - B. Math. III

Assignment 7 — 1st Semester 2023-2024

Due date: October 30, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

For $f \in L^1[-\pi, \pi]$, the Fourier coefficients of f are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nx \, dx, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \sin nx \, dx.$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the *Fourier series* of f, and we write

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

1. (10 points) Assume that f is a 2π-periodic function integrable on $[-\pi, \pi]$ and that f is of bounded variation on $[x_0 - \delta, x_0 + \delta]$. Show that the Fourier series of f at x_0 converges to $\frac{1}{2}(f(x_0^+) + f(x_0^-))$.

2. (10 points) (a) (5 points) With justification, provide an example of a function of bounded variation on $[-\pi, \pi]$ which does not satisfy any Lipschitz condition.

   (b) (5 points) With justification, provide an example of a function g that satisfies the Lipschitz condition at zero, that is, $|g(x) - g(0)| \leq |x|$ but g is not of bounded variation on any neighborhood of zero.

   (Thus the Dirichlet-Jordan test does not contain the Dini-Lipschitz test and vice versa.)

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3. (25 points) (Gibbs phenomenon) Let f be a 2π-periodic function whose values on $[-\pi, \pi]$ are given by:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } x \in \{0, \pi\} \end{cases}$$

   (a) (5 points) Show that

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}, \forall x \in \mathbb{R}.$$

   (b) (5 points) Show that the partial sums s_N are given by,

$$s_N(x) = \frac{2}{\pi} \int_0^x \frac{\sin 2Nt}{\sin t} dt.$$

   (c) (10 points) Find the points of local maxima and minima of s_N in the interval $(0, \pi)$.

   (d) (5 points) Prove that amongst the points of local maxima for s_N, the maximum value is attained at $\frac{\pi}{2N}$.

   (e) (5 points) Interprete $s_N(\frac{\pi}{2N})$ as a Riemann sum and prove that

$$\lim_{N \to \infty} s_N\left(\frac{\pi}{2N}\right) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

   (Hint: Check Exercise 11.19 from Apostol.)

4. (15 points) Consider the Fourier series (in exponential form) generated by a 2π-periodic function $f \in C^1(\mathbb{R})$. say

$$f(x) \sim \sum_{n \in \mathbb{Z}} \alpha_n e^{2\pi i n x}.$$

   (a) (10 points) Prove that the series $\sum_{n \in \mathbb{Z}} n^2 |\alpha_n|^2$ converges, and deduce that $\sum_{n \in \mathbb{Z}} |\alpha_n|$ converges.

   (b) (5 points) Deduce that the series $\sum_{n \in \mathbb{Z}} \alpha_n e^{2\pi i n x}$ converges uniformly to a continuous sum function g. Then prove that $f = g$.

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