Function Spaces - B. Math. III

Assignment 8 — 1st Semester 2023-2024

Due date: November 30, 2023

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

For $f \in L^1[-\pi, \pi]$, the Fourier coefficients of f are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nx \, dx, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \sin nx \, dx.$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the *Fourier series* of f, and we write

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

1. (10 points) Let $f \in L^1(\mathbb{R})$. For $r > 0$, let $f_r : \mathbb{R} \to \mathbb{R}$ be the function defined by,

$$f_r(x) = \frac{1}{2r} \int_{x-r}^{x+r} f(x) \, dx.$$

   (a) (5 points) Show that f_r is continuous for every $r > 0$ and $\|f_r\|_1 \leq \|f\|_1$.

   (b) (5 points) Show that $\lim_{r \to 0} \|f_r - f\|_1 = 0$.

2. (10 points) Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$ such that $f(x) \neq \pm\infty$. Then x is called a Lebesgue point for f if

$$\lim_{r \to 0} \frac{1}{r} \int_{x}^{x+r} |f(t) - f(x)| \, dt = 0.$$

   (a) (5 points) Show that if x is a Lebesgue point for f, then the function $x \mapsto \int_{-\infty}^{x} f(t) \, dt$ is differentiable at x, and its derivative at x is $f(x)$.

   (b) (5 points) Show that each point of continuity of f is a Lebesgue point for f.

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3. (20 points) Let $f \in L^1[-\pi, \pi]$.

   (a) (5 points) If $f \in L^2[-\pi, \pi]$, show that the series
   $$\sum_{N=1}^{\infty} \frac{|a_N| + |b_N|}{N}$$
   converges.

   (b) (10 points) If f is a 2π-periodic function in $C^1(\mathbb{R})$, then show that
   $$\|f - s_N\|_\infty = o\left(\frac{1}{\sqrt{N}}\right).$$
   (In other words, the error term for uniform approximation of f via s_N declines like "little oh" of $\frac{1}{\sqrt{N}}$.)

   (c) (5 points) If f is bounded, show that $|s_N(x)| = O(\ln N)$. (Hint: $\int_1^x \frac{1}{t}\, dt = \ln x$ and use estimates for the Dirichlet kernel.)

4. (10 points) Prove the *Féjer theorem*: Let f be a 2π-periodic function on $\mathbb{R}$ and $f \in L^1[-\pi, \pi]$. For $x \in [-\pi, \pi)$, assume that the limits $f(x^-), f(x^+)$ exist. Then show that
   $$\lim_{N\to\infty} \sigma_N(x) = \frac{f(x^+) + f(x^-)}{2}.$$

5. (10 points) Let $f \in L^1[-\pi, \pi]$ and x be a Lebesgue point for f (as defined in Problem 2 above). Show that
   $$\lim_{N\to\infty} \sigma_N(x) = f(x).$$

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