Indian Statistical Institute

B3: Complex Analysis (HW - 1) Due Date: 23 January, 2024 Instructor : Jaydeb Sarkar

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(i) $U \subseteq \mathbb{C}/\mathbb{R}^2$ open. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $Hol(U) = \{f : U \to \mathbb{C} \text{ holonorphic }\}$. (v) $\mathbb{D} = B_1(0)$. (vi) $\mathcal{D}$ = a domain in $\mathbb{C}$.

(1) Prove that $||z| - |w|| \leq |z - w|$ for all $z, w \in \mathbb{C}$.

(2) Prove that $Hol(\mathbb{D})$ is a vector space over $\mathbb{C}$. Is $Hol(\mathbb{D})$ finite dimensional?

(3) Characterize all (a) real linear maps from $\mathbb{C}$ to $\mathbb{C}$, (b) complex linear maps from $\mathbb{C}$ to $\mathbb{C}$.

(4) Let $f \in Hol(\mathcal{D})$. If $|f|$ is constant on $\mathcal{D}$, then prove that f is constant on $\mathcal{D}$.

(5) Let $u = \frac{xy}{x^2+y^2}$ for all $(x, y) \neq (0, 0)$ and $u(0, 0) = 0$, and $v(x, y) = 0$ for all (x, y). Prove that the C-R equation holds for the pair (u, v) at $(0, 0)$. Prove that (however) $f = u + iv$ is not holomorphic at $(0, 0)$.
[What's wrong! - Find out and explain.]

(6) Let $\bar{U} = \{z \in \mathbb{C} : \bar{z} \in U\}$, and let $f \in Hol(U)$. Prove that $F \in Hol(\bar{U})$ where $F(z) = \overline{f(\bar{z})}$. Compute F'.
[Question: Is $\bar{U}$ an open set?]

(7) (C-R Equation in Polar coordinate): Let $x = r\cos\theta$, $y = r\sin\theta$. Prove that the C-R equation('s) for $f = u + iv$ in polar coordinates is given by:

$$ru_r = v_\theta, \quad rv_r = -u_\theta.$$

(8) Let $f = u + iv \in Hol(\mathbb{C})$ (that is, f is an entire function). Suppose $h : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $u = h \circ v$. Prove that f is constant on $\mathbb{C}$.

(9) Write ∂ and $\bar{\partial}$ in polar coordinates. [Do NOT turn the solution.]

(10) Let $p \in \mathbb{C}[z, \bar{z}]$ (so typically, $p = \sum_{l,m \geq 0} \alpha_{lm} z^l \bar{z}^m$, $\alpha_{lm} \in \mathbb{C}$). Prove that $p \in \mathbb{C}[z]$ if and only if $\bar{\partial} p = 0$.

(11) Let f on $\mathbb{C}$ be a $\mathbb{C}$-valued function, and let $\alpha \in \mathbb{C}$. What do you mean by "$f(z) \to \infty$ as $z \to \alpha$"?

(12) Find an upper bound of

$$\Big| \int_{C_2(0)} \frac{e^z}{z^2+1} dz \Big|.$$

(13) True or False:

$$\int_{C_1(0)} \bar{z} dz = \int_{C_1(0)} \frac{1}{z} dz.$$

(14) Let γ = the line joining $-i$ to $1 + 2i$. Compute

$$\int_\gamma \Im z \, dz.$$