Indian Statistical Institute

B3: Complex Analysis (HW - 2) Due Date: 02 February, 2024 Instructor : Jaydeb Sarkar

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(i) $U \subseteq \mathbb{C}/\mathbb{R}^2$ open. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $Hol(U) = \{f : U \to \mathbb{C}$ holonorphic $\}$. (v) $\mathbb{D} = B_1(0)$. (vi) $\mathcal{D} =$ a domain in $\mathbb{C}$.

(1) Let $M > 0$ and let $|a_n| \leq M$ for all $n \geq 0$. Prove or disprove that the radius of convergence of $\sum_{n\geq 0} a_n z^n$ is ≥ 1.

(2) Determine the region of convergence of the following series: (i) $\sum_{n=1}^{\infty}(z-1)^n$, (ii) $\sum_{n=3}^{\infty}(\log n)^{\frac{n}{2}} z^n$, (iii) $\sum_{n=1}^{N} 2^n \log n\, z^n$, N is a fixed integer, (iv) $\sum_{p \text{ prime}} z^p$.

(3) Determine the power series expansion and the radius of convergence of f at z_0 where: (i) $f(z) = \frac{1}{4-z}$ and $z_0 = i$, (ii) $f(z) = \frac{z^2}{4-z}$ and $z_0 = i$, (iii) $f(z) = \frac{1}{z}$ and $z_0 = 1 - i$, (iv) $f(z) = 1 + z + z^2 + z^3$ and $z_0 = i$, (v) $f(z) = \frac{1}{(z+1)(z+2}$ at $z_0 = 0$.

(4) Prove using power series that $e^{-z} = \frac{1}{e^z}$.

(5) Prove that $f'(z) = \frac{1}{z^2+1}$, where

$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots .$$

(6) Let $\overline{B_R(0)} \subseteq U$, $f \in Hol(U)$ and let $r < R$. Prove that (Cauchy theorem!!)

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta = \frac{1}{\pi R^2}\int\int_{B_R(0)} f(x+iy)dydx.$$

(7) Let γ be a simple closed curve enclosing 0. Compute

$$\int_\gamma \frac{e^z + e^{-z}}{2z^4}dz.$$

(8) Compute

$$(a) \int_{C_1(0)} \frac{\sin z}{z}dz, (ii) \int_{C_1(0)} \frac{\cos z}{z}dz, (c) \int_{C_1(0)} \frac{\cos(z^2)}{z}dz.$$

(9) Compute

$$(a) \int_{C_1(-1)} \frac{z^2+1}{z^2-1}dz, (ii) \int_{C_3(0)} \frac{\exp z}{z^3}dz, (c) \int_{C_1(0)} \exp(z^2)dz.$$

(10) Prove that an entire function whose real part is nonpositive is constant.

(11) Let $f \in Hol(U)$ and let $\overline{B_r(z_0)} \subseteq U$. Prove that

$$\int_{C_r(z_0)} \frac{f(z)}{(z-z_0)^2}dz = \int_{C_r(z_0)} \frac{f'(z)}{(z-z_0)}dz.$$

(12) Suppose f is an entire function and

$$f(z) = f(z+1) = f(z+i).$$

for all $z \in C$. Prove that f is a constant function.

(13) Let $f \in Hol(\mathbb{C})$ and $f(z) \to \infty$ as $z \to \infty$. Prove that $f(z_0) = 0$ for some $z_0 \in \mathbb{C}$. [What does it mean - " $f(z) \to \infty$ as $z \to \infty$"?]

(14) If $f, g \in Hol(\mathbb{C})$ and $|f(z) \leq |g(z)| \neq 0$ for all $z \in \mathbb{C}$, then prove that $f = cg$ for some scalar c.

(15) True/False? (i) Let $f \in Hol(\mathbb{C})$ and $|f'(z)| \leq |z|$ for all z. Then $f(z) = a + bz^2$ for some $a, b \in \mathbb{C}$. (ii) Let $f \in Hol(\mathbb{C})$, and let f is constant on $C_1(0)$. Then f is constant. (iii) $\exists\ f \in Hol(\mathbb{D})$ such that $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^2}$ for all $n \geq 1$.

(16) Prove that the image of a nonconstant entire function is dense in $\mathbb{C}$.