Indian Statistical Institute

B3: Complex Analysis (HW - 3) Due Date: 28 March, 2024 Instructor : Jaydeb Sarkar

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(i) $U \subseteq \mathbb{C}/\mathbb{R}^2$ open. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $Hol(U) = \{f : U \to \mathbb{C}$ holonorphic $\}$. (v) $\mathbb{D} = B_1(0)$. (vi) $\mathcal{D}$ = a domain in $\mathbb{C}$.

(1) Compute $Res[\frac{f'}{f}; z_0]$ in each of the following cases: (i) z_0 is a zero of f of order n; (ii) z_0 is a pole of f of order n.

(2) Find the residues of the following functions at the isolated singular points in the extended complex plain $\mathbb{C}^*$ $(= \mathbb{C} \cup \{\infty\})$:

$$(i)\ \frac{z^2 + z - 1}{z^2(z-1)}, \quad (ii)\ \frac{1}{z^3 - z^5}.$$

(3) Let f has a pole of order n at z_0. Prove that

$$Res[f; z_0] = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \Big((z - z_0)^n f(z) \Big).$$

(4) Compute

$$(i) \int_{C_2(1)} \frac{dz}{z^4 + 4}. \quad (ii) \int_{C_1(0)} \frac{dz}{z^2 \sin z}. \quad (iii) \int_{C_2(0)} \frac{\sin z \, dz}{4z^2 - \pi^2} dz.$$

(5) Suppose that f has a simple pole at z_0 and let g be analytic in an open set containing z_0. Prove that

$$Res[fg; z_0] = g(z_0) Res[f; z_0].$$

(6) Let f and g be analytic at z_0. If $f(z_0) \neq 0$ and g has a simple zero at z_0, then show that

$$Res[\frac{f}{g}; z_0] = \frac{f(z_0)}{g'(z_0)}.$$

(7) If $p(z) = \Pi_{i=1}^n (z - \alpha_i)$, then compute $\frac{p'}{p}$. Also prove that

$$\frac{1}{2\pi i} \int_\gamma \frac{p'(z)}{p(z)} dz = \sum_{i=1}^n W(\gamma; \alpha_i),$$

where γ is a smooth closed path such that none of the roots of p lie on γ.