Indian Statistical Institute

B3: Complex Analysis (HW - 4) Due Date: Take your time Instructor : Jaydeb Sarkar

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(i) $U \subseteq \mathbb{C}/\mathbb{R}^2$ open. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $Hol(U) = \{f : U \to \mathbb{C}$ holonorphic $\}$. (v) $\mathbb{D} = B_1(0)$. (vi) $\mathcal{D}$ = a domain in $\mathbb{C}$. (vi)$\mathbb{H}$ = the upper half plane.

(1) Evaluate the integrals

$$(i) \int_{-\infty}^{\infty} \frac{x^4}{1+x^6} dx, \ (ii) \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx, \ (iii) \int_{-\infty}^{\infty} \frac{x^2}{(a^2+x^2)(b^2+x^2)} dx \ (a, b \in \mathbb{R}),$$

by the residue method.

(2) Prove that

$$Res[f; \infty] = Res\Big[\frac{1}{z^2} f\Big(\frac{1}{z}\Big); 0\Big].$$

(Recall: Let $f \in Hol(\{z : |z| > r\})$, for some $r > 0$, and let ∞ be an isolated singularity of f. If $r > R$, then the residue of f at infinity is defined to be $Res[f; \infty] = \frac{1}{2\pi i} \int_{-C_r(0)} f(z) dz$.)

(3) Does there exist a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?

(4) Suppose $f : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic. By considering the function

$$g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)},$$

prove that

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \le |f(z)| \le \frac{|f(0)| + |z|}{1 + |f(0)||z|} \qquad (z \in \mathbb{D}).$$

(5) Let $f : \mathbb{H} \to \overline{\mathbb{D}}$ be holomorphic and let $f(i) = 0$. Prove that

$$|f(z)| \le \Big|\frac{z - i}{z + i}\Big| \qquad (z \in \mathbb{H}).$$

(6) The pseudo-hyperbolic distance between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) = \Big|\frac{z - w}{1 - \bar{w} z}\Big|.$$

Prove that if $f : \mathbb{D} \to \bar{\mathbb{D}}$ is holomorphic, then $\rho(f(z), f(w)) \le \rho(z, w)$ for all $z, w \in \mathbb{D}$. Moreover, prove that if $f \in Aut(\mathbb{D})$, then f preserves the pseudo-hyperbolic distance, that is, $\rho(f(z), f(w)) = \rho(z, w)$ for all $z, w \in \mathbb{D}$.

(7) Prove that for every z and w in $\mathbb{D}$ there is $f \in Aut(\mathbb{D})$ with $f(z) = w$.