$\boxed{6/3/24}$ Recall that using Schwarz's lemma, we determined the group of all conformal autos. of the open unit disc $D := B(0,1)$. It turned out to be the group

$$PSU_{1,1}(\mathbb{C}) = \left\{ g \in SL_2(\mathbb{C}) \;\middle|\; \bar{g}^t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \Big/ \{\pm I\}$$

$$= \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \text{ and } |a|^2 - |b|^2 = 1 \right\} \Big/ \{\pm I\}.$$

Indeed, if $f : D \to D$ is a conformal auto. and $f(a) = 0$, then $f \circ \varphi_{-a}$ is a rotation $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$ by Schwarz. $\left(\text{here } \varphi_b = \begin{pmatrix} 1 & -b \\ -\bar{b} & 1 \end{pmatrix}^t \right)$

$$\Rightarrow f = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \varphi_a \in SU_{1,1}.$$

Using $\mathbb{H} \to D$; $z \overset{\theta}{\longmapsto} \dfrac{z - i}{1 - iz}$ conf. isom.,

we get $\operatorname{Aut}(\mathbb{H}) = \theta^{-1} \cdot \operatorname{Aut}(D) \cdot \theta = SL_2(\mathbb{R}) / \{\pm I\}$
$= PSL_2(\mathbb{R})$.

In fact, $\theta^{-1} f \theta = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} a & \cancel{b} \\ \bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} x + v & u + y \\ u - y & x - v \end{bmatrix}$

where $a = x + iy$, $b = u + iv$ and $|a|^2 - |b|^2 =$

- Spaces covered by $\mathbb{H}$ would then be $\mathbb{H}/\Gamma$ for some group Γ acting properly discontinuously on $\mathbb{H}$ $\iff$ discrete subgroup of $PSL_2(\mathbb{R})$.

- Uniformization theorem tells us that upto conformal equivalence, there are only three R.S's: $\mathbb{C}$, $\mathbb{CP}^1$ & $\mathbb{H}$.

  Looking at $\mathrm{Aut}\,\mathbb{C}$, $\mathrm{Aut}\,\mathbb{CP}^1$ and $\mathrm{Aut}\,\mathbb{H}$,

  it turns out that $\mathbb{C}$ covers only $\mathbb{C}$, $\mathbb{C}^\times$ & $\mathbb{C}/\text{lattice in } \mathbb{C}$; $\mathbb{CP}^1$ covers only itself, and $\mathbb{H}$ covers all the other R.S's

  (in particular, all R.S's of genus ≥ 2).

- We'll explicitly construct ~~the~~ holomorphic covering map $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0,1\}$.

  If this is done, then Little Picard follows immediately by lifting because $\mathbb{H}$ is ~~the~~ univ. cover

  [any holo. $f : \mathbb{C} \to \mathbb{C} \setminus 2$ pts. can be wlog $f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ & $\exists\, \tilde{f}$ holo: $\mathbb{C} \to \mathbb{H} \cong \mathbb{D}$ $\Rightarrow \tilde{f}$ const. by Liouville $\Rightarrow f$ const].

$$
\begin{tikzcd}
 & \mathbb{H} \arrow[d, "\lambda"] \\
\mathbb{C} \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & \mathbb{C} \setminus \{0,1\}
\end{tikzcd}
$$

- We'll also give other pfs. of Little Picard & Big Picard, proving on the way, many other interesting things.

A peek into elliptic functions

- A fn f on $\mathbb{C}$ is *doubly periodic* if $\exists\, \omega_1, \omega_2 \in \mathbb{C}$, $\neq 0$ & $\omega_1/\omega_2 \notin \mathbb{R}$ s.t. $f(z+\omega_i) = f(z)$ $\forall i = 1,2$ & $\forall z$ (wherever f is defined).

(If $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$, then ω_1, ω_2 are integer multiples of the same period; if $\frac{\omega_1}{\omega_2} \in \mathbb{R} \setminus \mathbb{Q}$, then f has arbitrarily small periods which'd mean it is const. in case f analytic (Ex.))

- If every period of f is in $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \Omega$, one calls the pair $\{\omega_1, \omega_2\}$ *fundamental*. This'll determine a network of //grams. — the period //grams.

- $\{\omega_1, \omega_2\}$ with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ and $\{\omega_1', \omega_2'\}$ with $\frac{\omega_1'}{\omega_2'} \notin \mathbb{R}$ are *equivalent* if $\Omega = \Omega'$ ($\Leftrightarrow \exists\, \gamma \in GL_2(\mathbb{Z})$ s.t. $\gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$)

- A fn f on $\mathbb{C}$ is called *elliptic* if it is doubly periodic and ~~is~~ meromorphic.

If f is elliptic, it has fundamental pairs of periods (Ex.) (Choose ω_1 one period closest to 0 & any smallest $\arg \geq 0$ at distance $|\omega_1|$ etc.)

- f elliptic with <u>no</u> poles in some period //gm P, then f is const. (Liouville ∵ f is bdd. on $\overline{P}$ & ∴ on $\mathbb{C}$)

— do — <u>no zeros</u> — " — (apply to $\frac{1}{f}$).

- A *cell* is a period //gm P s.t. $\nexists$ zeros, poles on ∂P. (Can get cells by translations)

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- f elliptic & P any cell $\Rightarrow \int_{\partial P} f = 0$ (cancellation due to periodicity)

<u>Cor</u>. Sum of the residues of an elliptic fn. at its poles in any period llgm is 0. [Cauchy's residue Th applied to a cell & above fact]

- P any cell & f elliptic, then $Z(f) = P(f)$ inside P

$\left(\frac{1}{2\pi i} \int_{\partial P} \frac{f'}{f} = Z(f) - P(f);\right.$ but $\frac{f'}{f}$ elliptic with same periods as f & $\therefore$ LHS $= 0 \Big)$

<u>Construction</u> (due to Weierstrass 2nd order pole & we don't discuss Jacobi's 2 simple poles)

The idea is simple (akin to $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$).

Fix $\omega_1, \omega_2 \in \mathbb{C}$ with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ & let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

Note that the double series $\sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \frac{1}{\omega^\alpha}$ cvges absolutely iff $\alpha > 2$.

(Ex.).

$\therefore$ For $\alpha > 2$ & $R > 0$, $\sum_{|\omega| \geq R} \frac{1}{(z-\omega)^\alpha}$ cvges absolutely & uniformly in $|z| \leq R$

(just. bound $\frac{1}{|z-\omega|^\alpha}$ as $\leq \frac{M}{|\omega|^\alpha}$ for $|\omega| > R$)

<u>Warning</u> Hence we <u>can't</u> define an elliptic fn as $\sum_{\omega \in \Omega} \frac{1}{(z-\omega)^2}$.

However

- $f(z) \stackrel{\text{def}}{=} \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$ is elliptic with periods ω_1, ω_2

  and a pole of order 3 at 0 (and at each period in Ω)

  ~~[works in view of abs. & unif. convergence in $|z| \leq R$; for $|z| > R$]~~

The $\wp$-f. is obtained by integrating the above f. term-by-term.

Def $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$

- $\wp$ is analytic except for a double pole at each $\omega \in \Omega$

  $\wp(z) = \wp(-z)$

  $\wp'(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$ is periodic as well.

- $\wp$ is doubly periodic (for Ω)

  ($\therefore$ $\wp(z+\omega) - \wp(z)$ is const $\forall \omega$ but taking $z = -\frac{\omega}{2}$ & evenness, deduce $\wp$ is periodic for Ω)

- If $r = \min\{|\omega| : \omega \neq 0 \text{ in } \Omega\}$, then for $0 < |z| < r$,

  $\wp(z) = \frac{1}{z^2} + \sum_{n \geq 1} (2n+1) G_{2n+2} z^{2n}$ where $G_k = \sum_{\omega \neq 0} \frac{1}{\omega^k}$ $\forall k \geq 3$.

  (Ex.)

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By forming a linear comb of powers of $\wp, \wp'$ such that the pole at $z=0$ is eliminated, we have:

$$\wp'(z)^2 = 4\wp^3(z) - 60 g_4 \wp - 140 g_6$$

$\wp' = -\frac{2}{z^3} + \cdots \Rightarrow \wp'(z)^2 = \frac{4}{z^6} \cancel{- \frac{24 g_4}{z^2}} \; 80 g_6 + \cdots$;

$\wp = \frac{1}{z^2} + \cdots \Rightarrow \wp^3 = \frac{1}{z^6} + \cancel{\frac{9 g_4}{z^2}} \; \cancel{+15} g_6 + \cdots$

$\Rightarrow \wp'^2 - 4\wp^3 = -\frac{60}{z^2} g_4 + \cdots \quad 140 g_6 + \cdots$

$\Rightarrow \wp'^2 - 4\wp^3 + 60 g_4 \wp = -140 g_6 + \cdots$

LHS has no poles & $\therefore$ is const which must be $-140 g_6$

- $e_1 = \wp(\frac{\omega_1}{2})$, $e_2 = \wp(\frac{\omega_2}{2})$, $e_3 = \wp(\frac{\omega_1+\omega_2}{2})$ satisfy

$$\wp'^2 = 4\wp^3 - 60 g_4 - 140 g_6 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

$\wp'(\omega_1 - z) = \wp'(-z) = -\wp'(\cancel{z})$ but for $z = \frac{\omega_1}{2}$, $\wp'(\frac{\omega_1}{2}) = -\wp'(\frac{\omega_1}{2}) \Rightarrow \frac{\omega_1}{2}$ root of $\wp'(z)$ etc.

Also e_1, e_2, e_3 are distinct, & disc $g_2^3 - 27 g_3^2 \neq 0$

where $g_2 = 60 g_4$, $g_3 = 140 g_6$.

Important [it follows $\because$ $\wp$ takes e_k twice, the same value is taken 4 times, but order of $\wp$ is 2]

- $\lambda(\tau) \stackrel{d}{=} \frac{e_3 - e_2}{e_1 - e_2}$ with $\tau = \frac{\omega_2}{\omega_1}$ is well-defined

($\because e_k$ is homog. of order $= -2$)

Choose ω_1, ω_2 st. $\tau \in \mathfrak{h}$.

Note λ is analytic on $\mathfrak{h}$ & takes values in $\mathbb{C} \setminus \{0, 1\}$.

If $\gamma \in \Gamma(2)$, easy to see $\lambda(\gamma\tau) = \lambda(\tau)$

$\Gamma / \Gamma(2) \cong S_3$ & $\Gamma(2) \cong F_2$ free

$$\left.\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1 \\ \omega_1' &= c\omega_2 + d\omega_1 \end{aligned}\right\} \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$$

$$\Rightarrow \lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \quad \lambda\left(-\frac{1}{\tau}\right) = 1 - \lambda(\tau).$$

Thus λ is the holomorphic covering map from $\mathfrak{h}$ to $\mathbb{C} \setminus \{0,1\}$ which affords Picard's Little Th. by the lifting property. [Details to be carefully seen from Ahlfors's §3 of last chapter 8]

Remark: If $g_2, g_3 \in \mathbb{C}$ satisfy $g_2^3 - 27 g_3^2 \neq 0$, $\exists$ a lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ s.t. $g_2 = g_2(\Omega)$, $g_3 = g_3(\Omega)$.

Enter André Bloch & Edmund Landau

[Littlewood on Bloch's Th. §?? : One of the queerest things in math — the pf. itself is crazy]

Th. (Bloch) Let E be the open unit disc $B(0,1)$. If $f \in \mathcal{O}(\overline{E})$ & $f'(0) = 1$, then

$f(E) \supseteq$ a disc of radius $\frac{3}{2} - \sqrt{2} > \frac{1}{12}$.

[Warning: The pt. $f(0)$ may not be the center of an image disc as in the Th.; e.g. $f_n(z) = \frac{e^{nz} - 1}{n} = z + \cdots$ omits the value $-\frac{1}{n}$]

Cor. f holo. on a domain G, and $f'(c) \neq 0$ at some point $c \in G \Rightarrow f(G) \supseteq$ discs of every radius $\frac{1}{12} s |f'(c)|$ $\forall\, 0 < s < d(c, \partial G)$.

Pf. WLOG $c = 0$. For $0 < s < d(c, \partial G)$, $\overline{B(0,s)} \subseteq G$

$\therefore g(z) \overset{\text{d}}{=} \frac{f(sz)}{s f'(0)} \in \mathcal{O}(\overline{E})$ & $|g'(0)| = 1$. So Bloch's Th.

$\Rightarrow g(E) \supseteq$ discs of radius $\frac{1}{12}$. As $f(B(0,s)) = s|f'(0)|\, g(E)$

Q.E.D.