SubCor. If $f \in \mathcal{O}(\mathbb{C})$ is not constant, then $f(\mathbb{C}) \supseteq$ discs of every radius.

Towards Little Picard Th. (Later, will do Bloch's Pf).

Want to show f entire which omits $a \neq b$ is const.

Wlog $a = 0$, $b = 1$ $\because$ apply to $\dfrac{f(z) - a}{b - a}$.

Basic idea is if G is a simply-connected domain, then the units of $\mathcal{O}(G)$ have logs & square roots in $\mathcal{O}(G)$.

Lemma Let $G \subseteq \mathbb{C}$ be simp-conn. domain & let $f \in \mathcal{O}(G)$ s.t. $-1, 1 \notin f(G)$. Then $\exists\, F \in \mathcal{O}(G)$ st $f = \cos F$.

Pf $1 - f^2$ has no zeros $\in G \Rightarrow \exists\, g \in \mathcal{O}(G)$ with $g^2 = 1 - f^2$.

$\therefore (f + ig)(f - ig) \equiv 1$. $\Rightarrow f \pm ig$ have no zeros

& $\therefore f + ig = e^{iF}$ with $F \in \mathcal{O}(G)$.

$\therefore f - ig = e^{-iF}$ & $\therefore f = \cos F$.

Prop. $G \subseteq \mathbb{C}$ simp-conn., $f \in \mathcal{O}(G)$, $0, 1 \notin f(G)$.

Then $\exists\, g \in \mathcal{O}(G)$ s.t. $f(z) = \dfrac{1 + \cos(\pi \cos \pi g(z))}{2}$

(by lemma above applied to $2f - 1$)

For any g satisfying this, $g(G) \not\supseteq$ a disc of radius 1.

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Pf First part is ok $\left(2f-1 = \cos \pi F \text{ \& } \operatorname{Im}(F) \cap \mathbb{Z} = \phi \right.$
$\Rightarrow F = \cos \pi g$ etc.

very rough Idea of 2nd part is as follows.

The set $A \stackrel{d}{=} \left\{ m \pm \frac{i \log(n+\sqrt{n^2-1})}{\pi} \;\middle|\; m \in \mathbb{Z}, n \geq 1 \right\}$
(it is a rectangular grid)

Satisfies: (i) $A \cap g(G) = \phi$, (ii) each disc of radius 1 intersects A

Let $a = b \pm i\pi^{-1}\log(\ell + \sqrt{\ell^2-1}) \in A$.

Claim $g(\mathbb{C}) \not\ni a$. Indeed, $\cos \pi a = \dfrac{e^{i\pi a} + e^{-i\pi a}}{2} = (-1)^b \ell$.

$\therefore$ $\cos \pi(\cos \pi a) = \pm 1 \notin g(\mathbb{C})$.

Now, the pts of A are vertices of rectangular grid.

'Length' $=$ (x-length) $= 1$.

Height $= \log(n+1 + \sqrt{(n+1)^2-1}) - \log(n + \sqrt{n^2-1})$

$< \pi$ (check!)

$\therefore$ $\forall w \in \mathbb{C}$, $\exists a \in A$ st $|\mathrm{Re}\, a - \mathrm{Re}\, w| \leq \frac{1}{2}$

& $|\mathrm{Im}\, a - \mathrm{Im}\, w| < \frac{1}{2}$

$\therefore$ Each unit disc $\cap A \neq \emptyset$.

Pf of Little Picard

If $f \in Hol(\mathbb{C})$, $f(\mathbb{C}) \not\ni 0,1 \Rightarrow$
$f = \dfrac{1 + \cos(\pi \cos \pi g)}{2}$ with $g \in Hol(\mathbb{C})$,

and $g(\mathbb{C}) \not\supset$ any disc of radius 1.

But, Bloch $\Rightarrow$ f contains discs of every radius if f is nonconst.

Note $f(z) = z + e^z$ has no fixed pts.

Applens of Picard

- $f: \mathbb{C} \to \mathbb{C}$ holo. $\Rightarrow f \circ f$ has a fixed pt unless $f(z) = z + b$, $b \neq 0$

Pf If $f \circ f$ has no fixed pts., then

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$ is holo. & avoids 0, 1.

$\therefore \exists\, c \in \mathbb{C} \setminus \{0,1\}$ st $\forall z \in \mathbb{C}$, $f(f(z)) - z = c(f(z) - z)$.

$\therefore \quad f'(z)\left(f'(f(z)) - c\right) = 1 - c$.

As $c \neq 1$, f' has no zeroes & $f'(f(z))$ is never equal to c.

$\therefore f' \circ f$ omits 0 and $c \neq 0 \Rightarrow f' \circ f$ const $\Rightarrow f'$ const $\Rightarrow f(z) = az + b$

f has no fixed pts $\Rightarrow a = 1$, $b \neq 0$.

• f, g Mero. on $\mathbb{C}$ & $f^n + g^n = 1$ with $n \geq 3$

$\Rightarrow$ either f, g are const

or they have common poles.

Pf. Suppose $P(f) \cap P(g) = \emptyset$. But $f^n + g^n = 1 \Rightarrow P(f) = P(g)$.

$\therefore P(f) = \emptyset = P(g)$ i.e. f, g holo. Let $g \not\equiv 0$.

As $Z(f) \cap Z(g) = \emptyset$, $\frac{f}{g} \in \text{Mero}(\mathbb{C})$ & takes a value $a \in \mathbb{C}$ at $w \in \mathbb{C}$ $\Leftrightarrow f(w) = a\, g(w)$.

Now $1 = \prod_{r=1}^{n} (f - \zeta_r g)$ where $X^n + 1 = \prod_1^n (X - \zeta_r)$

$\therefore \frac{f}{g}$ ~~dis~~ omits ζ_r $(1 \leq r \leq n)$. As $n \geq 3$,

$\frac{f}{g}$ ~~mero~~ mero. & omits 3 values. (h mero. & omits a, b, c $\Rightarrow \frac{1}{h-a}$ holo. & omits $\frac{1}{b-a}, \frac{1}{c-a}$)

$\Rightarrow \frac{f}{g}$ const $c \neq \zeta_r$

$\therefore g^n (c^n + 1) = 1 \Rightarrow g, f$ consts.

Ex. • Little Picard $\Leftrightarrow$ $f, g \in \mathcal{O}(\mathbb{C})$ & $1 = e^f + e^g \Rightarrow f, g$ const.

• $f, g, h \in \mathcal{O}(\mathbb{C})$ & $h = e^f + e^g \Rightarrow h$ has either no zeros or ∞ly many zeros in $\mathbb{C}$.
(apply Little Picard to $g - f$)

• $h \in \mathbb{C}[X]$ of deg $> 0 \Rightarrow h e^f$ assumes all values.
(tricky!)

(Online) 9/3/24

Recall we're trying to prove $f \in \mathcal{O}(\bar{E})$, $f'(0) = 1 \Rightarrow$

$f(E) \supseteq$ discs of radius $\frac{3}{2} - \sqrt{2}$ where $E = B(0,1)$.

Lemma 1 $G \subseteq \mathbb{C}$ bounded domain, $f : \bar{G} \to \mathbb{C}$ cont.,

$f|_G$ is an open (this is automatic if $f \in \mathcal{O}(G)$!) map.

Let $a \in G$ be s.t. $s := \min_{z \in \partial G} |f(z) - f(a)| > 0$.

Then $f(G) \supseteq$ disc $B(f(a); s)$.

Pf. As $\partial(f(G))$ is cpt., $\exists\, w_0 \in \partial(f(G))$ s.t.

$$d(\partial f(G)), f(a)) = |w_0 - f(a)|.$$

As $\bar{G}$ is cpt., $\exists \{z_n\} \in G$ s.t. $\mathrm{Lt}\, f(z_n) = w_0$

and $z_0 := \mathrm{Lt}\, z_n \in \bar{G}$.

$\therefore f(z_0) = w_0$ as f is cont. on $\bar{G}$

As $f|_G$ is open and $w_0 \in \partial(f(G))$, $z_0 \notin G$

ie. $z_0 \in \partial G$.

$$\therefore |w_0 - f(a)| \geq \min_{z \in \partial G} |f(z) - f(a)| = s.$$

So $d(\partial f(G), f(a)) \geq s$.

Hence $B(f(a); s) \subseteq f(G)$

We apply Lemma 1 to holomorphic f's. as follows.

Lemma 2 — Let $V = B(a;r)$ with $r > 0$ & let $f \in \mathcal{O}(\overline{V})$, be non-constant. If $\|f'\|_V \leq 2|f'(a)|$, then

$$f(V) \supseteq B(f(a);\ (3-2\sqrt{2})r|f'(a)|) \qquad \left[\begin{array}{l}\text{Note}\\ 3-2\sqrt{2} > \frac{1}{6}\end{array}\right]$$
$$= B(f(a);R) \text{ say.}$$

Pf. Wlog, $a = 0 = f(a)$. Consider

$$A(z) \stackrel{d}{=} f(z) - f'(0)z = \int_{[0,z]} (f'(w) - f'(0))\,dw.$$

$$\therefore |A(z)| \leq \int_0^1 |f'(zt) - f'(0)|\,|z|\,dt.$$

For $v \in V$, Cauchy's integral formula gives

$$f'(v) - f'(0) = \frac{v}{2\pi i}\int_{\partial V} \frac{f'(w)\,dw}{w(w-v)}.$$

$$\therefore |f'(v) - f'(0)| \leq \frac{|v|}{r-|v|}\|f'\|_V \quad \text{as } V = B(0;r).$$

Hence $$|A(z)| \leq \int_0^1 \frac{|zt|\,\|f'\|_V}{r-|zt|}|z|\,dt \leq \underbrace{\frac{|z|^2}{r-|z|}\|f'\|_V}_{\circledast}$$

Let $0 < \rho < r$. Then $\overbrace{|f(z) - f'(0)z|}^{|A(z)| =} \geq |f'(0)|\rho - |f(z)|$
$\forall z$ s.t. $|z| = \rho$.

By hypothesis, $\|f'\|_V \leq 2|f'(0)|$. So, $\circledast \Rightarrow |f(z)| \geq \left(\rho - \frac{\rho^2}{r-\rho}\right)|f'(0)|$

But $\rho - \frac{\rho^2}{r-\rho}$ assumes its max. value $(3-2\sqrt{2})r$ at $\rho_0 = \left(1-\frac{\sqrt{2}}{2}\right)r \in (0,r)$

$\therefore |f(z)| \geq \underbrace{(3-2\sqrt{2})r|f'(0)|}_{=R}$ $\forall |z| = \rho_0$. Use Lemma 1 with $G = B(0;\rho_0)$ to deduce $B(0;R) \subseteq f(G) \subseteq f(V)$ as $G \subseteq V$.

Pf. of Bloch's Thm.

don't even assume $|f'(0)| = 1$

If $f \in \mathcal{O}(\overline{E})$ non-const., consider the fn. $|f'(z)|(1-|z|)$, which is cont. on $\overline{E}$. Suppose its max. is obtained at a point $p \in E$, and the value $|f'(p)|(1-|p|) = M$.

We claim that $f(E) \supseteq B(f(p); (\frac{3}{2}-\sqrt{2})M)$

and that $(\frac{3}{2}-\sqrt{2})M > \frac{1}{\sqrt{2}}|f'(0)|$.

Indeed, if $t = \dfrac{1-|p|}{2}$, then $M \stackrel{d}{=} |f'(p)|(1-|p|)$
$= 2t\,|f'(p)|$

and $B(p;t) \subseteq E$ and $1-|z| \geq t$
$\forall z \in B(p;t)$

$(\because |z| = |z-p+p| \leq |z-p| + |p|$
$< t + |p| = 1-t)$

$\therefore \forall z \in B(p;t)$,

$|f'(z)|(1-|z|)$

$\leq 2t\,|f'(p)| \leq 2(1-|z|)\,|f'(p)|$

$\Rightarrow |f'(z)| \leq 2|f'(p)|$.

$\therefore$ By lemma 2, $B(f(p);R) \subseteq f(E)$,

where $R = (3-2\sqrt{2})t\,|f'(p)|$

$= \dfrac{3-2\sqrt{2}}{2}M > \dfrac{1}{12}M \geq \dfrac{1}{12}|f'(0)|$

Remark: The best Bloch constant doesnot seem to be known.