Global Cauchy Theorem

Winding numbers: If D is a disc with center z_0 and ∂D denotes the path which surrounds the boundary of D once in the anticlockwise orientation, then an elementary calculation shows that $\frac{1}{2\pi i}\int\limits_{\partial D}\frac{dz}{z-z_0}=1$. More generally, if γ goes around the boundary n times in the anticlockwise direction, then this integral equals $+n$, while if γ goes around the boundary n times in the clockwise direction, then the integral equals $-n$. This motivates the following:

Definition: If γ is a closed path and z_0 is a point outside $Im(\gamma)$, then the winding number $W(\gamma, z_0)$ is defined by

$$W(\gamma, z_0) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{dz}{z-z_0}.$$

Intuitively, this counts (with a sign for the direction), the net number of times γ goes around z_0. For this to be correct, the following lemma better be correct:

Lemma : $W(\gamma, z_0)$ is an integer.

Proof: Let $\gamma : [a, b] \to \mathbb{C}$. Define $F : [a, b] \to \mathbb{C}$ by $F(t) = \int\limits_a^t \frac{\gamma'(t)}{\gamma(t)-z_0}$.

Since γ is piecewise continuously differentiable, it follows that F is continuous on $[a, b]$ and $F'(t) = \frac{\gamma'(t)}{\gamma(t)-z_0}$ for all but finitely many points (where γ is not differentiable) in the interval $[a, b]$. Now define $G : [a, b] \to \mathbb{C}$ by $G(t) = \frac{e^{F(t)}}{\gamma(t)-z_0}$. Logarithmic differentiation yields $\frac{G'(t)}{G(t)} = F'(t) - \frac{\gamma'(t)}{\gamma(t)-z_0} = 0$ at all but finitely many points. Thus $G'(t) = 0$ at all these points. So G is a piecewise constant function. But G is continuous. So G is a constant function. In particular, $G(a) = G(b)$. That is, $\frac{e^{F(a)}}{\gamma(a)-z_0} = \frac{e^{F(b)}}{\gamma(b)-z_0}$. But we have $\gamma(a) = \gamma(b)$. Also, from its definition $F(a) = 0,\ F(b) = 2\pi i W(\gamma, z_0)$. Hence we get $e^{2\pi i W(\gamma,z_0)} = 1$. Thus $W(\gamma, z_0)$ is an integer. □

Exercise : If γ is a closed path in $\mathbb{C}$ then show that the function $z \mapsto W(\gamma, z)$ from $\mathbb{C}\backslash Im(\gamma)$ into $\mathbb{Z}$ is continuous. Hence conclude that $W(\gamma, z)$ is a constant on each connected component of $\mathbb{C}\backslash Im(\gamma)$. Also show that $W(\gamma, z) = 0$ for z in the unbounded component of $\mathbb{C}\backslash Im(\gamma)$ [Hint: Fix $z_0 \in \mathbb{C}\backslash Im(\gamma)$. Since $Im(\gamma)$ is compact, the distance of z_0 from points on

$Im(\gamma)$ is bounded away from 0. That is, $\exists\, r > 0$ such that $|\gamma(t) - z_n| \geq r \;\forall\, t$. It follows that if z is sufficiently close to z_0 then $|\gamma(t) - z \geq r/2$. Hence bound the absolute difference between the two integrands defining $W(\gamma, z)$ and $W(\gamma, z_0)$.]

Definition: We shall say that two paths $\gamma : [a, b] \to \Omega$ and $\eta : [a, b] \to \Omega$ are *close together* if they have the same initial point, same end point (i.e., $\gamma(a) = \eta(a), \gamma(b) = \eta(b)$) and there is a partition $a = a_1 < a_2 < \ldots < a_n = b$ and closed discs $D_1, \ldots, D_{n-1} \subseteq \Omega$ such that $\gamma([a_i, a_{i+1}]) \subseteq D_i, \quad \eta([a_i, a_{i+1}]) \subseteq D_i$ for $1 \leq i < n$.

Lemma : Let γ, η be two paths in Ω which are close together. Then for any holomorphic function $f : \Omega \to \mathbb{C}$, we have $\int\limits_{\gamma} f = \int\limits_{\eta} f$.

Proof: With notations as in the definition of "close together", we have a primitive g_i of f on $D_i,\ 1 \leq i < n$. Write γ_i for $\gamma|_{[a_i, a_{i+1}]}, \eta_i$ for $\eta|_{[a_i, a_{i+1}]}, 1 \leq i < n$. Also put $z_i = \gamma(a_i),\ w_i = \eta(a_i)$. Then $\int\limits_{\gamma} f = \sum\limits_{i=1}^{n-1} \int\limits_{\gamma_i} f = \sum\limits_{i=1}^{n-1} \int\limits_{\gamma_i} g_i' = \sum\limits_{i=1}^{n-1} (g_i(z_{i+1}) - g_i(z_i))$ and similarly $\int\limits_{\eta} f = \sum\limits_{i=1}^{n-1} (g_i(w_{i+1}) - g_i(w_i))$. But g_i and g_{i+1} are both primitives of f on the connected open set $D_i \cap D_{i+1}$. Hence $g_{i+1} - g_i$ is a constant on $D_i \cap D_{i+1}$. Also, $D_i \cap D_{i+1}$ contains both z_{i+1} and w_{i+1}. Therefore $g_{i+1}(w_{i+1}) - g_i(w_{i+1}) = g_{i+1}(z_{i+1}) - g_i(z_{i+1})$. That is $g_{i+1}(w_{i+1}) - g_{i+1}(z_{i+1}) = g_i(w_{i+1}) - g_i(z_{i+1})$ for $1 \leq i < n$.

Therefore,

$$\begin{aligned}
\int\limits_{\gamma} f - \int\limits_{n} f &= \sum_{i=1}^{n-1} ((g_i(z_{i+1}) - g_i(z_i)) - (g_i(w_{i+1}) - g_i(w_i))) \\
&= \sum_{i=1}^{n-1} ((g_i(z_{i+1}) - g_i(w_{i+1})) - (g_i(z_i) - g_i(w_i))) \\
&= \sum_{i=1}^{n-1} ((g_{i+1}(z_{i+1}) - g_{i+1}(w_{i+1})) - (g_i(z_i) - g_i(w_i))) \\
&= \sum_{i=1}^{n-1} (g_{i+1}(z_{i+1}) - g_i(z_i)) - \sum_{i=1}^{n-1} (g_{i+1}(w_{i+1}) - g_i(w_i))
\end{aligned}$$

Hence, by telescoping.

$$\begin{aligned}\int_\gamma f - \int_\eta f &= g_n(z_n) - g_1(z_1) - g_n(w_n) + g_1(w_1) \\ &= (g_n(z_n) - g_n(w_n)) - (g_1(z_1) - g_1(w_1))\end{aligned}$$

But $z_n = w_n =$ the common end point of γ and η and $z_1 = w_1 =$ the common initial point of γ and η. Hence $\int_\gamma f - \int_\eta f = 0$. □

Definition: Let $\gamma, \eta : [a,b] \to \Omega$ be two paths in Ω with common initial and end points. Then we say that γ and η are homotopic in Ω (with initial and end points held fixed) if there is a "continuous" one parameter family $\gamma_s : [a,b] \to \Omega,\quad 0 \le s \le 1$ such that $\gamma_s(a) = \gamma(a), \gamma_s(b) = \gamma(b)$ for all $s \in [0,1]$, and $\gamma_0 = \gamma, \gamma_1 = \eta$. (More precisely, the "continuity" requirement means that the "homotopy map" $(s,t) \mapsto \gamma_s(t)$ from $[0,1] \times [a,b]$ into Ω is continuous.)

Exercise : If γ, η are homotopic in Ω then use the uniform continuity of the homotopy map to show that there is an $\epsilon > 0$ such that for $s_1, s_2 \in [0,1]$ with $|s_1 - s_n| < \epsilon, \gamma_{s_1}$ and γ_{s_2} are close together. Hence conclude that there is a partition $0 = s_1 < s_2 < \ldots < s_n = 1$ such that $\gamma_{s_{i+1}}$ and γ_{s_i} are close together for $1 \le i < n$. Therefore, the above Lemma implies:

Homotopy Version of Cauchy's Theorem: If γ, η are homotopic paths in Ω then for any holomorphic $f : \Omega \to \mathbb{C}$, we have

$$\int_\gamma f = \int_\eta f.$$

A closed path γ in Ω is said to be *null-homotopic* if it is homotopic to a point (constant path). It is easy to see that the above theorem is equivalent to:

Alternative homotopy version of Cauchy's Theorem: If γ is a null-homotopic closed path in Ω then for any holomorphic $f : \Omega \to \mathbb{C}$, $\int_\gamma f = 0$.

Definition: Ω is said to be *simply connected* if every closed path in Ω is null homotopic in Ω (intuitively, this means that Ω has no holes). An immediate consequence of the theorem is:

Corollary: If Ω is simply connected, then for any closed path γ in Ω and any holomorphic $f : \Omega \to \mathbb{C}$, we have $\int\limits_\gamma f = 0$.

Corollary: If Ω is simply connected then any holomorphic $f : \Omega \to \mathbb{C}$ has a global primitive $g : \Omega \to \mathbb{C}$ such that $g' = f$.

Definition: Let γ and η be two paths in Ω with the same initial and end points ($\gamma(a) = \eta(a), \gamma(b) = \eta(b)$, where $\eta, \gamma : [a, b] \to \mathbb{C}$). We say that γ, η are homologous in Ω if $W(\gamma, z_0) = W(\eta, z_0) \forall z_0 \notin \Omega$. If γ is a closed path homologous to a point (constant path) then we say that γ is null-homologous.

For technical reasons, it is good to extend this definition as follows. We say that γ is a *chain* if it is a formal sum of finitely many paths. If γ is a formal sum of finitely many closed paths then we say that γ is a closed chain.

If $\gamma = \gamma_1 + \ldots + \gamma_n$, then $Im(\gamma) \stackrel{\text{def}}{=} \bigcup\limits_{i=1}^{n} Im(\gamma_i))$. If $z_0 \notin Im(\gamma)$, we define $W(\gamma, z_0) := \sum\limits_{i=1}^{n} W(\gamma_i, z_0)$. If each γ_i is a path in Ω we say that γ is a *chain in* Ω. For a chain $\gamma = \gamma_1 + \ldots + \gamma_n$ in Ω and $f : \Omega \to \mathbb{C}$, we define $\int\limits_\gamma f := \sum\limits_{i=1}^{n} \int\limits_{\gamma_i} f$. Two chains γ, η in Ω are called homologous in Ω if $W(\gamma, z_0) = W(\eta, z_0) \ \forall \ z_0 \notin \Omega$. A closed chain γ in Ω is called null-homologous in Ω if $W(\gamma, z_0) = 0 \ \forall \ z_0 \notin \Omega$.

With this definition, the most general version of Cauchy's fundamental theorem is:

Global Cauchy Theorem: If γ is a null-homologous closed chain in Ω then $\int\limits_\gamma f = 0$ for all holomorphic f on Ω. Equivalently, if γ, η are homologous closed chains in Ω then $\int\limits_\gamma f = \int\limits_\eta f$ for all holomorphic f on Ω.

This is the most general version of Cauchy in the sense that if a closed chain γ is not null-homologous in Ω then there is a holomorphic function f on Ω such that $\int\limits_\gamma f \neq 0$ (namely $f(z) = \frac{1}{z - z_0}$ for a suitable $z_0 \notin \Omega$).

To prove Cauchy's Global Theorem, we need the following two lemmas.

We shall say that a path is *rectangular* if it is a concatenation of horizontal and vertical line segments.

Lemma : If γ is a path in Ω then there is a rectangular path η in Ω such that γ and η are close together in Ω. Consequently (by the earlier Lemma on equality of integrals over close paths) γ and η are homologous, and $\int_\gamma f = \int_\eta f$ for any holomorphic f on Ω.

In view of this Lemma, to prove the Global Cauchy Theorem, it is enough to prove it for "rectangular" closed chains (i.e, formal sums of rectangular closed paths).

Proof of Lemma : Given $\gamma : [a,b] \to \Omega$ take a partition $a = a_1 < a_n < \ldots < a_n = b$ of $[a,b]$ such that $\gamma([a_i, a_{i+1}]) \subseteq D_i$, a closed disc in Ω. Put $\gamma_i = \gamma|_{[a_i,a_{i+1}]}, z_i = \gamma(a_i)$. Take a rectangular path η_i lying inside D_i and joining z_i to z_{i+1} $(1 \le i < n)$. Let η be the concatenation of $\eta_1, \ldots, \eta_{n-1}$. This clearly works.

If $\gamma : [a,b] \to \mathbb{C}$ is a path and $a = a_1 < \ldots < a_n = b$, $\gamma_i = \gamma|_{[a_i,a_{i+1}]}$, $1 \le i < n$ then the chain $\gamma = \gamma_1 + \ldots + \gamma_{n-1}$ will be called a subdivision of γ. More generally, if $\eta = n_1 + \ldots + n_m$ is a chain, $n_{i1} + \ldots + n_{in}$ is a subdivision of the path n_i for each i, then the chain $\sum_{i,j} n_{ij}$ will be called a subdivision of the chain η. Clearly if the chain η' is a subdivision of the chain η then η, η' are homologous (in any domain Ω such that η, η' are in Ω).

Lemma (Artin) : If γ is a rectangular closed chain in Ω which is null-homologous in Ω then there exist rectangles $R_1, \ldots R_N$ in Ω and integers $\alpha_1, \ldots, \alpha_N$ such that the rectangular closed chain $\eta = \sum_{i=1}^{N} d_i \partial R_i$ is a subdivision of γ. (Here ∂R_i is the closed path traversing the boundary of R_i anticlockwise).

Sketch of Proof: Draw all the lines which contain one of the line-segments constituting γ. These are (finitely many) horizontal and vertical lines. They partition the complex plane into finitely many regions, some of which are rectangles and others unbounded. If R is one of these rectangles then the interior R^0 is inside a connected component of $Im(\gamma)$, and hence the winding number $W(\gamma, \cdot)$ is a constant, say α_R, on R^0. If $\alpha_R \neq 0$ for some R then $R^0 \subseteq \Omega$ (since $W(\gamma, \cdot)$ is zero in the complement of Ω). Hence it is easy to see that the closed chain $\eta = \sum_{R:\alpha_R \neq 0} \alpha_R \cdot \partial R$ is a subdivision of γ. $\square$.

Now, the two previous lemmas together show that: if γ is a null-homologous closed chain in Ω then there are rectangles $R_1, \ldots, R_N$ in Ω and integers $\alpha_1, \ldots, \alpha_N$ such that $\int\limits_\gamma f = \int\limits_{\sum \alpha_i \partial R_i} = \sum\limits_{i=1}^{N} \alpha_i \int\limits_{\partial R_i} f = 0$, proving Cauchy's Global Theorem.

One application of Cauchy's Global Theorem is to reduce the calculation of integrals (of holomorphic functions) over complicated paths to those over simple paths. Namely, we have:

Theorem: Let γ be a null-homologous closed chain in Ω. Let $z_1, \ldots, z_N$ be finitely many distinct points in Ω and let $D_1, \ldots, D_N$ be pairwise disjoint closed discs in Ω with centres $z_1, \ldots, z_N$. Assume γ does not pass through any of the points z_i. Put $m_i = W(\gamma, z_i)\ \ 1 \leq i \leq n$. Then γ is homologous in $\Omega \backslash \{z_1, \ldots, z_N\}$ to the chain $\sum\limits_{i=1}^{N} m_i \partial D_i$. Hence, for any holomorphic function $f : \Omega \setminus \{z_1, \ldots, z_N\} \to \mathbb{C}$, we have

$$\int\limits_\gamma f = \sum_{i=1}^{N} m_i \int\limits_{\partial D_i} f.$$

Proof : If $\alpha \notin \Omega$ then $W(\gamma, \alpha) = 0$ and $W(\partial D_i, \alpha) = 0$. Hence $W(\sum m_i \partial D_i, \alpha) = 0 = W(\gamma, \alpha)$. On the other hand if $\alpha = z_i$ for some i then $W(\gamma, \alpha) = m_i$, while $W(\partial D_j, \alpha) = \delta_{ij}$. Hence $W(\sum m_j \partial D_j, \alpha) = \sum m_j\ \delta_{ij} = m_i = W(\gamma, \alpha)$. Thus $W(\gamma, \alpha) = W(\sum m_i \partial D_i, \alpha)$ for all α outside $\Omega \backslash \{z_1, \cdots, z_N\}$. this proves the first statement. The second statement follows from the global Cauchy theorem. □

Another consequence is:

The Global version of Cauchy's integral formula: Let γ be a null-homologous closed chain in Ω. Let $z_0 \in \Omega$ be such that γ does not pass through z_0. Then for any holomorphic $f : \Omega \to \mathbb{C}$, we have $\frac{1}{2\pi i} \int\limits_\gamma \frac{f(z)}{z - z_0} = W(\gamma, z_0) f(z_0)$.

Proof: In view of the definition of $W(\gamma, z_0)$, this formula may be written as $\frac{1}{2\pi i} \int\limits_\gamma g(z) dz = 0$ where $g : \Omega \to \mathbb{C}$ is defined by $g(z) = \frac{f(z) - f(z_0}{z - z_0}$ if $z \neq z_0,\ g(z_0) = f'(z_0)$. Since γ is null-homologous and g is (analytic and hence) holomorphic, this follows from Global Cauchy Theorem. □