Zero set of holomorphic functions on domains

The analogue of the Fundamental Theorem of Algebra for a general $f \in \mathcal{O}(\Omega)$ is that if $f^{-1}\{0\} \neq \emptyset$, then it *must be a discrete set in* Ω or vanish on an entire component of Ω. This is made more precise in the following theorem.

Theorem 0.1 *Let Ω be a connected open set in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Then, the following are equivalent:*

a) $f \equiv 0$*;*

b) $\exists a_0 \in \Omega$ *such that* $f^{(k)}(a_0) = 0 \ \forall k \in \mathbb{N}$*;*

c) $f^{-1}\{0\}$ *has a limit point in* Ω*.*

Proof: (a) $\Rightarrow$ (c) is obvious.

Let us now show that (c) $\Rightarrow$ (b): Let a be a limit point of $f^{-1}\{0\}$ lying in Ω. Since f is continuous, $f(a) = 0$. Suppose $\exists N \in \mathbb{Z}_+$ such that

$$\begin{aligned} f^{(k)}(a) &= 0 \quad \forall k < N, \\ f^{(N)}(a) &\neq 0. \end{aligned}$$

Let $R > 0$ such that $\overline{D(a;R)} \subset \Omega$. Then, in this disc, we have the power-series development

$$f(z) = \sum_{n=N}^{\infty} c_n (z-a)^n \equiv (z-a)^N g(z), \quad (c_N \neq 0) \quad \forall\ z \in D(a;R). \tag{1}$$

The statements in (1) follow from our assumptions on $f^{(k)}$. In particular, we see that $g \in \mathcal{C}(D(a;R);\mathbb{C})$ and that $\exists r \in (0,R)$ such that $g(z) \neq 0 \ \forall z \in D(a;r)$. We see from (1) thus that

$$f(z) \neq 0 \quad \forall z \in D(a;r) \backslash \{a\},$$

which violates the fact that a is a limit of $f^{-1}\{0\}$. Our assumption about $\{f^{(k)}\}_{k \in \mathbb{N}}$ must hence be wrong. Thus (c) $\Rightarrow$ (b).

It now remains to show that (b) $\Rightarrow$ (a). This is where we use the connectedness of Ω. Define

$$\begin{aligned} A &= \{z \in \Omega : f^{(k)}(z) = 0 \ \forall k \in \mathbb{N}\} \\ &= \bigcap_{k \in \mathbb{N}} [f^{(k)}]^{-1}\{0\}; \end{aligned}$$

the continuity of $f^{(k)}$ for each $k \in \mathbb{N}$ establishes that A is closed in Ω. $A \neq \emptyset$ by (b). Finally, pick $z_0 \in A$. Let $\rho > 0$ such that $\overline{D(z_0; \rho)} \subset \Omega$. Then, f has the power-series development

$$f(z) = \sum_{k \in \mathbb{N}} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad \forall z \in D(z_0; \rho).$$

Consequently, $f|_{D(z_0;\rho)} \equiv 0$, i.e. $D(z_0; \rho) \subset A$. Since z_0 was an arbitrary point in A, we conclude that A is open.

Thus A is a non-empty set that is both Ω-open and Ω-closed. As Ω is connected, $A = \Omega$. This establishes (a).