Further additions and clarifications of some things we discussed in class yesterday (5th February 2024) and earlier:

Remember that we are following Ahlfors in our discussions. Recall that our paths are piecewise differentiable. Recall also that I had defined (in the previous week) the index of a point with respect to a loop (or winding number of the loop around that point), and proved some of its properties yesterday. These were as follows:

Definition. If γ is a closed path and z_0 is a point outside $Im(\gamma)$, then the winding number $W(\gamma, z_0)$ is defined by

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}.$$

Intuitively, this counts (with a sign for the direction), the net number of times γ goes around z_0. This makes sense in view of the following lemma in section 2.1 of chapter 4 of Ahlfors:

Lemma. The number $W(\gamma, z_0)$ is an integer. (Ahlfors calls it the index of z_0 w.r.t. γ, and denotes it as $n(\gamma, z_0)$).

Proof. Let $\gamma : [a, b] \to \mathbb{C}$. Define $F : [a, b] \to \mathbb{C}$ by $F(t) = \int\limits_a^t \frac{\gamma'(t)}{\gamma(t)-z_0}$.
Since γ is piecewise continuously differentiable, it follows that F is continuous on $[a, b]$ and $F'(t) = \frac{\gamma'(t)}{\gamma(t)-z_0}$ for all but finitely many points (where γ is not differentiable) in the interval $[a, b]$. Now define $G : [a, b] \to \mathbb{C}$ by $G(t) = \frac{e^{F(t)}}{\gamma(t)-z_0}$. Logarithmic differentiation yields $\frac{G'(t)}{G(t)} = F'(t) - \frac{\gamma'(t)}{\gamma(t)-z_0} = 0$ at all but finitely many points. Thus $G'(t) = 0$ at all these points. So G is a piecewise constant function. But G is continuous. So G is a constant function. In particular, $G(a) = G(b)$. That is, $\frac{e^{F(a)}}{\gamma(a)-z_0} = \frac{e^{F(b)}}{\gamma(b)-z_0}$. But we have $\gamma(a) = \gamma(b)$. Also, from its definition $F(a) = 0,\ F(b) = 2\pi i W(\gamma, z_0)$. Hence we get $e^{2\pi i W(\gamma, z_0)} = 1$. Thus $W(\gamma, z_0)$ is an integer. $\square$

We proved yesterday (the next result in Ahlfors) that the winding number is constant on the two connected components. More precisely, *If γ is a closed path in $\mathbb{C}$, then the function $z \mapsto W(\gamma, z)$ from $\mathbb{C} \backslash Im(\gamma)$ into $\mathbb{Z}$ is continuous.*

This implies that $W(\gamma, z)$ is a constant on each connected component of $\mathbb{C} \backslash Im(\gamma)$. Also, $W(\gamma, z) = 0$ for z in the unbounded component of $\mathbb{C} \backslash Im(\gamma)$.

Idea of proof:- Fix $z_0 \in \mathbb{C} \backslash Im(\gamma)$. Since $Im(\gamma)$ is compact, the distance of z_0 from points on $Im(\gamma)$ is bounded away from 0. That is, $\exists \ r > 0$ such that $|\gamma(t) - z_n| \geq r \ \forall \ t$. It follows that if z is sufficiently close to z_0 then $|\gamma(t) - z \geq r/2$. Hence bound the absolute difference between the two integrands defining $W(\gamma, z)$ and $W(\gamma, z_0)$.

Yesterday, we had also discussed what would be the 'most general form' of Cauchy's theorem. Instead of looking for domains where integrals of analytic functions over ALL closed paths inside it are zero - which happens for simply connected domains, one looks at general domains and tries to find WHICH closed paths have the property of this vanishing integral. Call a closed path γ (or a formal concatenation of closed paths) to be 'null-homologous' if $W(\gamma, z_0) = 0$ for all z_0 outside $\bar{D}$. Then, here was the point of view I mentioned:

Let D be a domain in $\mathbb{C}$. Consider the vector space generated by all closed paths in D, and consider the quotient space $H_1(D)$ of the above by those elements which are null-homologous. Similarly, consider the vector space $H^1(D)$ of holomorphic functions on D quotiented by the subspace generated by those which are derivatives. Then, the map

$$(\gamma, f) \mapsto \int_\gamma f dz;$$

$$H_1(D) \times H^1(D) \to \mathbb{C}$$

is a non-degenerate bilinear pairing.

In other words, these two spaces are naturally duals of each other. In this form of Cauchy's theorem, we see a simple form of what is called the de Rham theorem.

I mentioned yesterday that concerning the local behaviour of analytic functions, aspects such as the 'order of a zero' or the notion of 'removable singularity' have not yet been discussed in Jaydeb's class. But, I stated the 'finite'

Taylor expansion for analytic function, without proving it.

This is theorem 8 of section 3 of chapter 4 of Ahlfors, and depends only on the fact that using Cauchy's integral formula, one can deduce that for a holomorphic function f in a domain D, and a point $z_0 \in D$. there is a holomorphic function f_1 which equals $\frac{f(z)-f(z_0)}{z-z_0}$ for $z \neq z_0$, and $f_1(z_0) = f'(z_0)$. This is repeated with f_1 in place of f etc. This is how one obtains the finite Taylor theorem (Theorem 8 mentioned above).

The point to note in the above theorem is that in the finite version

$$f(z) = \sum_{r=0}^{n-1} f^{(r)}(z_0)\frac{(z-z_0)^r}{r!} + f_n(z)(z-z_0)^n,$$

the 'error term' $f_n(z)$ is the analytic function given explicitly as

$$f_n(z) = \frac{1}{2i\pi}\int_\gamma \frac{f(w)dw}{(w-z_0)^n(w-z)}$$

where γ is any circle around z_0 such that the corresponding closed disc is contained in D. This enables us to observe that

$$|f_n(z)(z-z_0)^n| \leq \frac{M|z-z_0|^n}{R^{n-1}(R-|z-z_0|)}$$

if M is an upper bound for $|f(z)|$ on γ, and R is the radius of γ. Clearly, this tends to 0 uniformly in every closed disc $\{|z-z_0| \leq r\}$ with $r < R$. If we choose R to be arbitrarily close to the distance between z_0 and ∂D, we have the usual (infinite) Taylor series (chapter 5, section 1.2):

$$f(z) = \sum_{n\geq 0} f^{(n)}(z_0)\frac{(z-z_0)^n}{n!}$$

which is valid in the largest open disc centered at z_0 and contained in D.

The reason to recall the above facts about Taylor series etc., is to make some strong observations, which are as follows:

For a holomorphic function f (other than the zero function), we shall observe that (similar to polynomials) each zero z_0 has a 'finite multiplicity' in the sense that there is some $n > 0$ such that $f^{(n)}(z_0) \neq 0$. A word of warning here is that it may be true that $f^{(k)}(z_0) = 0$ for infinitely many k, nevertheless! The function $Sin(z)$ is an example.

The implication "*If $f \in Hol(D)$ satisfies, for some z_0, the property $f^{(n)}(z_0) = 0$ for all $n \geq 0$, then f must be the zero function on D*", follows by looking at the zero set of f and showing that it is also open, using the existence of the Taylor series above. The above properties are also equivalent to the zero set having a limit point in D.
The main result that we note from the above observation is that for each holomorphic function f on D (other than the zero function) and any $z_0 \in D$, there is a unique positive integer $n \geq 0$ such that $f^{(k)}(z_0) = 0$ for all $k < n$, and $f^{(n)}(z_0) \neq 0$. Thus, if $f(z_0) = 0$, one can write $f(z) = (z - z_0)^n g(z)$ for some $n \geq 1$, and some holomorphic function g on D such that $g(z_0) \neq 0$.
Note that the corresponding result for real functions on intervals is not true.