0.1 The logarithm:

Suppose we want to define $\log(z)$ to be the inverse of e^z, then if we write $\log(z) = u(z) + iv(z),\ u, v$ real-valued, then we have

$$\exp[u(z) + iv(z)] = z.$$

Suppose, we write $z = |z|e^{i\mathsf{Arg}(z)}$, where we choose the argument so that $\mathsf{Arg}(z) \in [-\pi, \pi)$. Then

$$|z| = e^{u(z)}|e^{iv(z)}| = e^{u(z)},$$

whence $u(z) = \log|z|$. But there is no unique choice for $v(z)$ since the above calculation actually reveals:

$$\{w \in \mathbb{C} :\ e^w = z\} = \{\log|z| + i(\mathsf{Arg}(z) + 2\pi k) : k \in \mathbb{Z}\}.$$

Suppose we make a *choice* of $k \in \mathbb{Z}$ to define $v(z)$ and set, for instance:

$$\log(z) := \log|z| + i\mathsf{Arg}(z)\ \forall z \in \mathbb{C} \setminus \{0\}, \tag{1}$$

we are led to the question: Is $\log \in \mathcal{O}(\mathbb{C} \setminus \{0\})$? Unfortunately, log defined in (1) is not even continuous. To see this, note that whereas, by the definition of Arg,

$$\log(-1) = -i\pi,$$

$$\begin{aligned}\lim_{t\to 0^+} \log(-1 + it) &= \lim_{t\to 0^+} \left\{\log\sqrt{1+t^2} + i\cos^{-1}\left(\frac{-1}{\sqrt{1+t^2}}\right)\right\} \\ &= +i\pi \neq \log(-1).\end{aligned}$$

This problem is resolved if we restrict Arg to take values in $(-\pi, \pi)$ (note the open interval). This amounts to restricting z to $G_0 := \mathbb{C} \setminus (-\infty, 0]$. Let the logarithm restricted to G_0 be denoted by Log, i.e.

$$\mathsf{Log}(z) := \log|z| + i\mathsf{Arg}(z) \quad \forall z \in G_0 := \mathbb{C} \setminus (-\infty, 0].$$

Log is known as the *the principal analytic branch of the logarithm.* Certainly $\mathsf{Log} \in \mathcal{C}(G_0; \mathbb{C})$, but why is it analytic? It turns out that continuity is the crucial property needed, from which analyticity follows in view of the following:

Exercise: Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open subsets. Let $f \in \mathcal{C}(\Omega_1; \mathbb{C}), g \in \mathcal{O}(\Omega_2)$, and $f(\Omega_1) \subset \Omega_2$. Suppose

$$g[f(z)] = z \quad \forall z \in \Omega_1,$$

and $g'(w) \neq 0 \forall w \in \Omega_2$. Then $f \in \mathcal{O}(\Omega_1)$ and

$$f'(z) = \frac{1}{g'[f(z)]} \quad \forall z \in \Omega_1.$$

Each choice of $v(z)$ in the definition of $\log(z)$ in (1) leads to a different inverse of e^z. These different choices of the logarithm are called the *branches* of the logarithm. We then have the following result.

Proposition 0.2 *Let $\Omega \subset \mathbb{C}$ be an open and connected set, and let $F, G \in \mathcal{O}(\Omega)$ be two analytic branches of the logarithm. Then, $\exists k_0 \in \mathbb{Z}$ such that $F(z) = G(z) + 2\pi i k_0$ $\forall z \in \Omega$.*

Proof: Let us define the function

$$\nu(z) := \frac{F(z) - \mathrm{G}(z)}{2\pi i} \quad \forall z \in \Omega.$$

Since $\{w \in \mathbb{C} : e^w = z\} = \{\log|z| + i(\mathsf{Arg}(z) + 2\pi k) : k \in \mathbb{Z}\}$, we see that $\nu(z) \in \mathbb{Z}$ $\forall z \in \Omega$. But ν is clearly continuous. Hence, as Ω is connected, $\exists k_0 \in \mathbb{Z}$ such that $\nu(\Omega) = \{k_0\}$. Hence

$$F(z) - G(z) = 2\pi i k_0 \quad \forall z \in \Omega.$$