**Arzela-Ascoli, Montel, Riemann Mapping Theorems**

**Remraks.**

For sets of functions defined on a domain $\Omega$ in $\mathbb{C}$, what is the correct topology to put so that the Bolzano-Weierstrass property holds for compact sets of functions? We will define a metric topology that accomplishes this. This can be done for families of functions which take values in a complete metric space.

Let $\Omega \subset \mathbb{C}$ be a domain. We can consider a so-called 'increasing compact exhaustion' $\{K_n\}_n$ of $\Omega$. More precisely, $K_t \subseteq K_{t+1}$ for all $t \geq 1$, and each compact subset $K \subset \Omega$ is contained in some $K_t$. Indeed, one may choose $K_t$ to be the subset of $z \in \Omega$ satisfying $|z| \leq t$ and $d(z, \delta\Omega) \geq 1/t$.
Let $\mathcal{F}_0$ be the family of all functions from $\Omega$ to a complete metric space $(X, d_0)$ where $d_0$ is a bounded metric (note that if $\delta$ is a metric, then $d_0(x,y) = \delta(x,1)/(1+\delta(x,y))$ is a bounded metric.
Define, for $f, g \in \mathcal{F}_0$, and $n \geq 1$,

$$d_n(f,g) := sup\{d_0(f(z), g(z) : z \in K_n\}.$$

Then $d(f,g) := \sum_{n\geq 1} 2^{-n} d_n(f,g)$ is a metric on $\mathcal{F}_0$.
In what follows, we specialize to $X = \mathbb{C}$; this is for simplicity in the case of the Arzela-Ascoli theorem, and is a necessity in case of families of complex holomorphic functions.

**Definition.** Let $\{f_n\}$ be a sequence of complex-valued functions on $\Omega$. Let $f$ be a complex-valued function on $\Omega$. We shall say that $f_n \to f$ *locally uniformly* if for each $z_0 \in \Omega$, there is a neighborhood $U$ of $z_0$ $(U \subseteq \Omega)$ such that $f_n(z) \to f(z)$ uniformly for $z \in U$.

It is trivial to check that $f_n \to f$ locally uniformly on $\Omega$ iff for every compact set $K \subseteq \Omega$, $f_n(z) \to f(z)$ uniformly for $z \in K$.
Note that the limit function may not be in the family we start with.

**Definition.** A family $\mathcal{F}$ of complex-valued functions on $\Gamma$ is said to be *normal* if every sequence $\{f_n\}$ has a subsequence which converges uniformly on compact subsets of $\Omega$ - this definition is due to Paul Montel.

**Exercise.** Show that a family $\mathcal{F}$ is normal iff it is relatively compact. (For this reason, sometimes one uses the word 'precompact' in place of 'normal'.

To point out that the above notion of local uniform convergence is the right one while dealing with holomorphic functions, we note:

**Theorem (Weierstrass).** If $\{f_n\}$ is a sequence of holomorphic functions on $\Omega$ such that $f_n \to f$ locally uniformly on $\Omega$, then $f$ is holomorphic on $\Omega$.

**Proof.** Let $R \subseteq \Omega$ be any rectangle. Then $f_n \to f$ uniformly on $R$. Hence $\int\limits_{\partial R} f_n \to \int\limits_{\partial R} f$. But by local Cauchy theorem, $\int\limits_{\partial R} f_n = 0 \; \forall \; n$. So $\int\limits_{\partial R} f = 0$. Hence by Morera's theorem, $f$ is holomorphic. □

In short, "local uniform" limits of holomorphic functions are holomorphic. This shows that local uniform convergence is the right notion of convergence for holomorphic functions. Another evidence for this is the following result, which says that "complex differentiation" is a continuous function on the space of holomorphic functions.

**Theorem.** If $f_n, f$ are holomorphic functions on a domain $\Omega$ such that $f_n \to f$ locally uniformly on $\Omega$ as $n \to \infty$, then $f_n' \to f'$ locally uniformly on $\Omega$.

**Proof.** Fix $z_0 \in \Omega$. Let $D$ be a closed disc contained in $\Omega$ with center $z_0$. Then we have the Cauchy integral formula $f(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(z)}{z-w} dz, \; w \in D^0$. Differentiating with respect to $w$, we get

$$f'(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(z)}{(z-w)^2} dz, \; w \in D^0.$$

Similarly, $f_n'(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f_n(z)}{(z-w)^2} dz, \; w \in D^0$. Now, let $E$ be a closed disc with centre $z_0$ such that $E \subseteq D^0$. As $f_n \to f$ locally uniformly, we have $\frac{f_n(z)}{(z-w)^2} \to \frac{f(z)}{(z-w)^2}$ uniformly for $z \in \partial D, \; w \in E$ (why ?). Integrating with respect to $z$, we get $f_n'(w) \to f'(w)$ uniformly for $w \in E$. □

**Exercise (Huwitz's theorem).** Using this theorem and the argument principle, prove that if $\{f_n\}$ is a sequence of holomorphic functions on $\Omega$ converging to $f$ locally uniformly on $\Omega$, and $\gamma$ is a null-homotopic closed path not passing through any zero of $f$ then $\exists N$ such that for $n \geq N$, the number of zeroes of $f_n$ (counting multiplicity) enclosed by $\gamma$ equals the number of zeroes of $f$ (counting multiplicity) enclosed by $\gamma$. Deduce that if all the $f_n$'s are non-vanishing on $\Omega$, then either $f \equiv 0$ or $f$ is non-vanishing on $\Omega$.

**Remarks.**

(i) The above results are valid only for holomorphic functions. They may be false for real analytic functions also. For example, the sequence of real analytic functions $sin(nx)$ has no subsequence which even converges pointwise; indeed, for any infinite sequence $n_1 < n_2 < n_3 < \cdots$ the set of points $x$ such that the limit of $Sin(n_k x)$ as $k \to \infty$ exists, has Lebsesgue measure 0.

(ii) Thus, there are some strong properties which would imply normalcy of a family that satisfies these properties. There are two notions; one addressing uniform boundedness as we vary in the family, and the other addressing continuity in a uniform way for the whole family. The latter notion called uniform equicontituity was defined by Ascoli.

A family $\mathcal{F}$ is said to be *locally* uniformly bounded (respectively locally uniformly equicontinuous) if any point $z_0 \in \Omega$ has a neighbourhood $U \subseteq \Omega$ on which $\mathcal{F}$ is uniformly bounded : $\exists\ c > 0 \ni |f(z)| \leq c \ \ \forall\ z \in U\ \forall\ f\ \in \mathcal{F}$ (respectively, on which $f$ is uniformly equicontinuous : $\forall\ z_0 \in U, \ \ \forall\ \ \epsilon > 0, \ \exists\ \delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon\ \forall\ z \in U, \ \forall\ f \in \mathcal{F}$.) Recall:

**Arzela(-Ascoli) Theorem :** If $\mathcal{F}$ is a locally uniformly bounded and locally uniformly equicontinuous family of (continuous) complex-valued functions on $\Omega$, then $\mathcal{F}$ is normal.

**Proof:** Fix a countable dense subset $D = \{w_j\} \subseteq \Omega$ (For instance, $D$ may consist of all points of $\Omega$ with real and imaginary parts in $\mathbb{Q}$.) We use Cantor's diagonal argument to show that any sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence $\{g_n\}$ which converges at all points of $D$. Then, we shall use local boundedness and local uniform equicontinuity of $\{g_n\}$ to conclude that $\{g_n\}$ converges locally uniformly on $\Omega$.
Now, let $\{f_n\}$ be a sequence in $\mathcal{F}$, and let $K \subset \Omega$ be compact. As $\{f_n\}$ is locally uniformly bounded, there is a subsequence $\{f_{n,1}\}_n$ such that $f_{n,1}(w_1)$ converges. Again, there is a subsequence $\{f_{n,2}\}_n$ of the sequence $\{f_{n,1}\}_n$

such that $f_{n,2}(w_2)$ converges. In this manner, get subsequences $\{f_{n,j+1}\}_n$ of $\{f_{n,j}\}_n$ for all $j \geq 1$ such that $f_{n,j}(w_k)$ converges for all $1 \leq k \leq j$.
Consider now the diagonal sequence $g_n = \{f_{n,n}\}_n$ which is clearly increasing, and clearly satisfies the property that $g_n(w_k)$ converges for all $k \geq 1$.
We show that $g_n$ converges uniformly on $K$. Give $\epsilon > 0$, let $\delta > 0$ be so that $|z - w| < \delta$ with $z, w \in K$ implies $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$. Now, for some $k$, we have $K \subset \bigcup_{i=1}^{k} B(w_i; \delta)$. Let $N > 0$ such that for all $m, n > N$ we have $|g_n(w_i) - g_m(w_i)| < \epsilon$ for all $1 \leq i \leq k$. Therefore, for $z \in K$ (say $z \in B(w_{i_0}; \delta)$ for some $i_0 \leq k$), we have for all $m, n > N$:

$$|g_n(z)-g_m(z)| \leq |g_n(z)-g_n(w_{i_0})|+|g_n(w_{i_0})-g_m(w_{i_0})|+|g_m(w_{i_0})-g_m(z)| < 3\epsilon.$$

Therefore, $\{g_n\}$ converges uniformly on $K$.
To deduce that uniform convergence holds on ALL compact sets, consider any increasing compact exhaustion $K_t \subseteq K_{t+1}$ for all $t \geq 1$ of $\Omega$, as above. Once again, use the diagonalization process to get a subsequence $\{g_{n,1}\}$ of $\{f_n\}$ which converges uniformly on $K_1$. Then get a subsequence $\{g_{n,2}\}$ of $\{g_{n,1}\}$ which converges uniformly on $K_2$, and so on. The diagonal sequence $\{g_{n,n}\}_n$ is a subsequence of $\{f_n\}$ which converges uniformly on each $K_t$. The sequence $\{K_t\}$ being an exhaustion of $\Omega$, this implies that the sequence $\{g_{n,n}\}_n$ is a subsequence of $\{f_n\}$ which converges uniformly on each compact subset of $\Omega$.

**Montel's Theorem.** If $\mathcal{F}$ is a locally uniformly bounded family of holomorphic function on $\Omega$, then $\mathcal{F}$ is normal (in other words, for holomorphic families, local uniform equicontinuity is automatic).

**Proof:** By the Arzela-Ascoli theorem, it is enough to show that $\mathcal{F}$ is locally uniformly equicontinuous. Fix $z_0 \in \Omega$. Let $D \subseteq \Omega$ be a closed disc with centre $z_0$. Then we have $f(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(w)}{w-z}$, $z \in D^0$, for each $f \in \mathcal{F}$.
Differentiating under the integral sign, we get

$$f'(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(w)}{(w-z)^2} \quad z \in D^0.$$

Therefore, if $D_1$ is a slightly smaller disc with centre $z_0$ then there is an absolute constant $c_1$ such that

$$|f'(z)| \leq c_1 \sup_{w \in D} |f(w)|, \; z \in D_1.$$

Now, as $\mathcal{F}$ is locally uniformly bounded there is a constant $c_2$ such $\sup\limits_{w\in D}|f(w)| \leq c_2$ for all $f \in \mathcal{F}$. Hence we get $|f'(z)| \leq c\ \forall\, z \in D_1,\ \forall\, f \in \mathcal{F}$ where $c = c_1c_2$. Therefore $|f(z) - f(z_0)| = |\int\limits_{z_0}^{z} f'(w)dw| \leq c|z - z_0|\ \ \forall z \in D_1,\ \forall f \in \mathcal{F}$. Then $\mathcal{F}$ is locally uniformly equicontinuous on $D_1$. □

If $\Omega_1,\ \Omega_2$ are open subsets of $\mathbb{C}$, then a map $f : \Omega_1 \to \Omega_2$ is said to be a biholomorphic isomorphism if $f$ is a bijection and both $f$ and $f^{-1}$ are holomorphic.

**Exercise.** Let $f : \Omega \to \mathbb{C}$ be holomorphic and injective. Then show that $f'(z) \neq 0\ \forall\ z \in \Omega$, and $f$ is a biholomorphic isomorphism between $\Omega$ and $f(\Omega)$.

**Exercise.** Let $\{f_n\}$ be a sequence of injective holomorphic functions from $\Omega$ to $\mathbb{C}$. Suppose $f_n \to f$ locally uniformly on $\Omega$. Then show that $f$ is either a constant function or injective. (*Hint.* Argument principle).

**Exercise.** If $\mathcal{F}$ is a normal family of holomorphic functions on $\Omega$ then show that the family $\mathcal{F}' := \{f'\ :\ f \in \Omega\}$ is again normal.

Let $\mathbb{D} = B(0;1)$ in what follows.

**Lemma.** Let $\Omega$ be a simply connected, connected, proper, open subset of $\mathbb{C}$. Then there is an injective holomorphic function $f : \Omega \to \mathbb{D}$.
**Proof.** Take $\alpha \in \mathbb{C}\backslash\Omega$. Since $\Omega$ is simply connected, there is a (global) primitive $g$ of $z \mapsto \frac{1}{z-\alpha}$ on $\Omega$. Then $e^{g(z)} = z-\alpha$. In consequence $g$ is injective on $\Omega$, and further, for any $z_0 \in \Omega$, $g$ does not assume the value $g(z_0) + 2\pi i$. Hence there is an $r > 0$ such that the *closed* disc $D$ with center $g(z_0) + 2\pi i$ and radius $r$ is disjoint from $g(\Omega)$. (Else there would be a sequence $\{z_n\}$ in $\Omega$ such that $g(z_n) \to g(z_0) + 2\pi i$, whence, exponentiating, $z_n \to z_0$ and hence $g(z_n) \to g(z_0)$. Hence $g(z_0) = g(z_0) + 2\pi i$ and $2\pi i = 0$, a contradiction.) Therefore the function $f(z) = \frac{1}{g(z)-g(z_0)-2\pi i}$ on $\Omega$ is injective, holomorphic and bounded. Scaling, $f$ can be made to map into $\mathbb{D}$. □

**Riemann Mapping Theorem.** Every simply connected, domain $\Omega$ properly contained in $\mathbb{C}$, is biholomorphically equivalent to $\mathbb{D}$, i.e., there is a biholomorphic isomorphism $g : \Omega \to \mathbb{D}$.
**Proof.** Without loss, we may assume that $0 \in \Omega$. By the lemma, there is a holomorphic injection $f : \Omega \to \mathbb{D}$. Since the Möbius group is transitive on $\mathbb{D}$, there is a Möbius map $\phi : \mathbb{D} \to \mathbb{D}$ such that $\phi$ takes $f(0)$ to 0. Then

$\tilde{f} = \phi \circ f$ is a holomorphic injection from $\Omega$ to $\mathbb{D}$ such that $\tilde{f}(0) = 0$.

Then the family $\mathcal{F} = \{f : \Omega \to \mathbb{D}$ such that $f$ is a holomorphic injection, $f(0) = 0\}$ is non-empty. By Montel's Theorem, $\mathcal{F}$ is a normal family. Therefore, there is an $f \in \mathcal{F}$ which maximises $|f'(0)|$ among all elements of $\mathcal{F}$. (If $\alpha = \sup_{f \in \mathcal{F}} |f'(0)|$, then there is a sequence $\{f_n\} \subseteq \mathcal{F}$ such that $|f_n'(0)| \to \alpha$. Since $\mathcal{F}$ is normal, we may replace $\{f_n\}$ by a suitable subsequence and assume $f_n \to f$ locally uniformly on $\Omega$. Then $f_n' \to f'$ locally uniformly. In particular, $f_n'(0) \to f'(0)$. Thus $|f_n'(0)| \to |f'(0)|$. Hence $|f'(0)| = \alpha > 0$. Since $f'(0) \neq 0$, $f$ is not a constant function. Therefore by Exercise 21, $f$ is injective. Since $f$ is non-constant, the maximum modulus principle implies that $f$ maps $\Omega$ into $\mathbb{D}$. So $f \in \mathcal{F}$).

Thus we may choose $f \in \mathcal{F}$ which maximises $|f'(0)|$. (This is a typical application of compactness: $\bar{\mathcal{F}}$ is compact and $f \mapsto |f'(0)|$ is a "continuous" function on $\bar{\mathcal{F}}$, hence is maximised somewhere in $\bar{\mathcal{F}}$. Then the argument principle comes in handy to show that the maximising "point" $f$ is actually in $\mathcal{F}$.).

We claim that this function $f$ maps $\Omega$ *onto* $\mathbb{D}$, and hence, is the required biholomorphic isomorphism between $\Omega$ and $\mathbb{D}$. Suppose otherwise. Then, there exists $\alpha \in \mathbb{D}$ such that $f$ does not take the value $\alpha$. Then there is a Möbius map $\varphi$ such that $\varphi(\alpha) = 0$. Hence $\varphi \circ f$ is a holomorphic injection from $\Omega$ into an open subset of $\mathbb{D}$ which does not contain 0. Since $\varphi \circ f$ is a holomorphic injection onto its image, this subset is also simply connected. Hence there is a holomorphic branch of the square root function on this subset ($z \mapsto \exp(\frac{1}{2} log z)$). Composing it with the function $\varphi \circ f$, we get a holomorphic function $z \mapsto \sqrt{\varphi(f(z))}$ from $\Omega$ into $\mathbb{D} \backslash \{0\}$. Let $\psi$ be a Möbius map sending $\sqrt{\phi(f(0))}$ to 0. Let $\tilde{f}(z) = \psi(\sqrt{\phi(f(z))}),\ z \in \Omega$. Then $\tilde{f}$ is a holomorphic map from $\Omega$ into $\mathbb{D}$, such that $\tilde{f}(0) = 0$. Also, $\tilde{f}$ is an injection ($\tilde{f}(z_1) = \tilde{f}(z_2) \Rightarrow \psi(\sqrt{\phi(f(z_1))}) = \psi(\sqrt{\phi f(z_2)}) \Rightarrow \sqrt{\phi(f(z_1))} = \sqrt{\phi(f(z_2))} \Rightarrow \phi(f(z_1)) = \phi(f(z_2)) \Rightarrow f(z_1) = f(z_2) \Rightarrow z_1 = z_2$.)

Then $\tilde{f} \in \mathcal{F}$. Let $S$ be the squaring function. Then we have $f = (\varphi^{-1} \circ S \circ \psi^{-1}) \circ \tilde{f}$. Hence $|f'(0)| = |(\varphi^{-1} \circ S \circ \psi^{-1})'(0)| \cdot |\tilde{f}'(0)|$. Therefore, $|(\varphi^{-1} \circ S \circ \psi^{-1})'(0)| \geq 1$ because $|f'(0)| \geq |\tilde{f}'(0)|$ as $\tilde{f} \in \mathcal{F}$. This contradicts the fact we deduced from Schwarz lemma which shows that a holomorphic map $h : D \to D$ such that $h(0) = 0$ and which is NOT 1-1, must satisfy $|h'(0)| < 1$. So, we have a contradiction, and $f$ must have been a surjection.