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Two Consequences of the Maximum Modulus Th.

[Stronger Versions due to I. Glicksberg – AMM 83('76) 186–187 of] Rouché's Thm.:–

Let f, g be meromorphic in a nbd. of $\overline{D(a,r)}$, and having no zeros or poles on the circle $C(a,r) =: C$.

If Z_f, P_f, Z_g, P_g count the zeros, poles inside C (counted with multiplicity), and

if $|f(z)+g(z)| < |f(z)| + |g(z)|$ on C,

then $Z_f - P_f = Z_g - P_g$.

Pf. On C, $\left|\frac{f}{g}+1\right| < \left|\frac{f}{g}\right|+1$.

This means $\frac{f}{g}$ can't take values in $[0,\infty)$.

So, $\frac{f}{g}$ maps C to $\mathbb{C} \setminus [0,\infty)$.

Thus, if $\log$ is a branch of the log on $\mathbb{C}\setminus[0,\infty)$,

then $\log\frac{f}{g}$ is a primitive for $\frac{(f/g)'}{(f/g)}$ in a nbd. of C.

So, $0 = \frac{1}{2\pi i}\int_C \frac{(f/g)'}{(f/g)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz - \frac{1}{2\pi i}\int_C \frac{g'}{g}\,dz$

$= (Z_f - P_f) - (Z_g - P_g)$.

Classical (weaker) version of Rouché has the stronger hypothesis that f, g are holo. & $|f(z)+g(z)| < |g(z)|$ on C, then $Z_f = Z_g$ inside C.

Example Find the number of zeros of $z^3+5z+3 =: f(z)$ in the annulus $\{1<|z|<3\}$.

Take $g(z) = z^3$

$$|f(z)-g(z)| = |5z+3| \le 15+3 = 18 \text{ for } |z|=3.$$

$$18 < 27 = 3^3 = |z^3| \text{ on } |z|=3.$$

$$\therefore |f(z)-g(z)| < |g(z)| \text{ on } |z|=3.$$

$\therefore Z_f = Z_g$ inside $|z|=3$ i.e. in $|z|<3$.

So, f has 3 zeros in $|z|<3$.

Again, consider $|f(z)-(5z+3)| = |z^3|$ on $|z|=1$.

$$\left.\begin{array}{l} 1 = |z^3| = |f(z)-(5z+3)| \\ \text{and } |5z+3| \geq 5|z|-3 = 2 > 1 \end{array}\right\} \text{on } |z|=1$$

So $|f(z)-(5z+3)| < |5z+3|$ on $|z|=1$.

$\therefore$ No. of zeros of f in $|z|<1$ is 1 $\because$ $5z+3$ has $-\frac{3}{5}$ as the 1 zero in $|z|<1$.

Also, on $|z|=1$, $|f(z)| = |5z+3+z^3|$

$$\geq |5z+3| - |z^3|$$

$$\geq 5|z| - 3 - |z|^3 = 1 > 0.$$

$\therefore$ No zeros of f on $|z|=1$.

So f has two zeros in $\{1<|z|<2\}$.

Schwarz Lemma D open unit disc.

Let $f \in \mathrm{Hol}(D)$ such that $f(0)=0$ and $|f(z)| \le 1 \ \forall z \in D$.

Then $|f(z)| \le |z| \ \forall z \in D$, and $|f'(0)| \le 1$.

Further, if either $|f'(0)|=1$ or $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$,

then $\exists \lambda \in S^1$ s.t. $f(z) \equiv \lambda z$.

Pf. The fn. $g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases}$

is holo. on D as it has a removable singularity at 0.

For $|z| \le r$ with $0<r<1$, the MMT $\Rightarrow |g(z)| \le \frac{1}{r}$.

$$\left(\because \sup_{|w|=r} |g(w)| = \sup_{|w|=r} \frac{|f(w)|}{r} \le \frac{1}{r} \right)$$

Now, let $r \to 1^-$, to get $|g(z)| \le 1 \ \forall z \in D$.

So $|f(z)| \le |z| \ \forall z \in D$ and $|f'(0)| = |g(0)| \le 1$.

If $|f(z_0)| = |z_0|$ for some $z_0 \in D$, $z_0 \neq 0$,

or if $|f'(0)| = 1$,

then $|g|$ assumes its max. inside D; this implies

by MMT for g, that $\exists \lambda \in S^1$ with $g(z) \equiv \lambda \ \forall z \in D$.

(note $g(z) \equiv$ const. $\forall z \in D$ $\Rightarrow$ $|$const.$| = 1$)

That is, $f(z) \equiv \lambda z$ (rotation).

Remarks

We make several remarks & also give applcns. which will indicate that Schwarz lemma is really a geometric statement.

In particular, we'll point out that ∃ a Riemannian metric on D s.t. any holo. ~~ff~~ from D to itself is distance-decreasing & ~~them~~ isometries can easily be determined
In other words, $|f'(a)|$ can be bdd. explicitly)

As $f: D \to D$ may not fix 0, we'll construct some map which takes ~~(0)~~ to 0 and a LFT.

Def $\forall\, a \in D$, we define a map $\varphi_a : D \to D$ (which turns out to be a conformal auto.!) as

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

Note
- $\varphi_0 = \mathrm{Id}$.
- φ_a is holo. on $|z| < \frac{1}{a}$ ($\therefore$ in a nbd. of $\overline{D}$)
- φ_a is bijective on D (indeed, $\varphi_a^{-1} = \varphi_{-a}$)
- $\varphi_a(\partial D) \subseteq \partial D$ $\left(\because\ \varphi_a(e^{i\theta}) = \left|\dfrac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right| = \left|\dfrac{e^{i\theta}-a}{\overline{e^{-i\theta}}-\bar{a}}\right| = 1\right)$
- $\varphi_a(0) = -a$
- $\varphi_a(a) = 0$.
- $\varphi_a'(0) = 1-|a|^2$ $\left(\because\ \varphi_a'(z) = \dfrac{1}{1-\bar{a}z} + \dfrac{\bar{a}(z-a)}{(1-\bar{a}z)^2}\right)$

& $\varphi_a'(a) = \dfrac{1}{1-|a|^2}$

- $ds \overset{d}{=} \dfrac{|dz|}{1-|z|^2}$, a Riemannian metric on $\mathbb{D}$,

& Schwarz's lemma $\Longleftrightarrow$ $\left\{ f : \mathbb{D} \to \mathbb{D} \text{ holo.} \Rightarrow f^*(ds) \le ds \right\}$

- How much does f contract?

$\to$ Let $f : \mathbb{D} \to \mathbb{C}$ be non-constt, holo, and $|f(z)| \le 1 \ \forall \ z \in \mathbb{D}$.

Then $$\frac{|f(0)| - |z|}{1 + |f(0) z|} \le |f(z)| \le \frac{|f(0)| + |z|}{1 - |f(0) z|} \quad \forall z \in \mathbb{D}$$

Pf. Wlog $f(0) \ne 0$; else Schwarz gives us.
Consider $g(z) \overset{d}{=} \dfrac{f(z) - f(0)}{1 - \overline{f(0)} f(z)}$; Now $g(0) = 0$.

Also $|g(z)| \le 1$ $(\because |f(z)| \le 1)$. By Schwarz for g, $|g(z)| \le |z| \ \forall z \in \mathbb{D}$

Simplify $|f(z) - f(0)| \le |z| \, |1 - \overline{f(0)} f(z)|$ to get

$$\frac{|f(0)| - |z|}{1 + |f(0)| |z|} \le |f(z)| \le \frac{|f(0)| + |z|}{1 - |f(0)| |z|} \quad \forall z \in \mathbb{D}.$$

Example $\nexists \ f \in Hol(\mathbb{D})$ s.t. $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) > \frac{7}{12}$.

Pf $$|f'(\tfrac{1}{2})| \le \left| \frac{1 - |f(1/2)|^2}{1 - (1/2)^2} \right| = \frac{7 \times 4}{16 \times 3} = \frac{7}{12}.$$

Cor. Let $f : D \to D$ be holo. bijection. Let $f(a) = 0$. Then, $\exists\, \lambda \in \mathbb{C}^\times$ s.t. $|\lambda| = 1$ and $f(z) = \lambda \varphi_a(z)$ $\forall\, z \in D$.

Pf. If g is the inverse f. of f on D, g is holo. $\because f'$ is nowhere zero ($\because f$ is 1–1).

So $(g \circ f)'(a) = g'(0) \cdot f'(a) = 1$.

By Schwarz–Pick applied to f & g, we get $|f'(a)| \le \dfrac{1}{1-|a|^2}$ & $|g'(0)| \le 1 - |a|^2$

& as the products of the LHS's is $= 1$, we have '$=$' in both the above statements.

So, by the last statement of prev. page, $f = \lambda \varphi_a$ on D, with $|\lambda| = 1$.

Remark More general (than Schwarz–Pick) fact: $f : D \to D$ holo. Then $\forall\, a, b \in D$,

$$\left| \frac{f(a) - f(b)}{1 - \overline{f(a)} f(b)} \right| \le \left| \frac{a - b}{1 - \bar{a} b} \right|$$

(taking $b \to a$, we get S–P)

Pf. $g(z) \stackrel{\text{def}}{=} \dfrac{\dfrac{f(z) - f(b)}{1 - \overline{f(b)} f(z)}}{\dfrac{z - b}{1 - \bar{b} z}}$ on $D \setminus \{b\}$ is holo. at 'b' also. By Schwarz, $|g| \le 1$ on D by MMT as $\dfrac{z-b}{1-\bar{b}z} \to 1$ as $|z| \to 1$ but $|f(z) - f(b)|$ ...

Now, let's solve the maximization problem of finding $f: D \to D$ such that $f(\alpha) = \beta$ and $|f'(\alpha)|$ is max. possible.

Note that if $f(\alpha) = \beta$, where $f: D \to D$ holo,

then $\varphi_\beta \circ f \circ \varphi_{-\alpha} : D \to D$ is holo. & takes 0 to 0.

Applying Schwarz's lemma to this f, we get

$$\left|\left(\varphi_\beta \circ f \circ \varphi_{-\alpha}\right)'(0)\right| \leq 1, \text{ which gives}$$

$$1 \geq \left|\varphi_\beta'\left((f \circ \varphi_{-\alpha})(0)\right) \cdot f'(\varphi_{-\alpha}(0)) \cdot \varphi_{-\alpha}'(0)\right|$$

$$= |\varphi_\beta'(\beta)| \cdot |f'(\alpha)| \cdot |\varphi_{-\alpha}'(0)|$$

$$= |f'(\alpha)| \cdot \frac{1 - |\alpha|^2}{1 - |\beta|^2}.$$

$$\therefore \boxed{|f'(\alpha)| \leq \frac{1 - |f(\alpha)|^2}{1 - |\alpha|^2}}$$

This is sometimes called the Schwarz-Pick lemma.

Moreover, '=' occurs iff $\left(\varphi_\beta \circ f \circ \varphi_{-\alpha}\right)(z) \equiv \lambda z$ $\forall z \in D$ for some $|\lambda| = 1$.

i.e. $f(w) \equiv \varphi_{-\beta}(\lambda \cdot \varphi_\alpha(w))$ $\forall w \in D$, for some $|\lambda| = 1$.
(Take $w = \varphi_{-\alpha}(z)$).