LIST OF PROBLEMS

(1) Let $P(z) = z + \sum_{k=2}^{n} a_k z^k$ be a complex polynomial of degree n, which is one-one in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Show that $|a_n| \leq (1/n)$.

(2) Assume that the complex numbers $a_2, a_3, \cdots$ are such that

$$\sum_{n=2}^{\infty} n|a_n| < 1.$$

Show that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

defines a function holomorphic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and that f is one-one in $\mathbb{D}$.

(3) Suppose $U \subset \mathbb{C}$ is open, f is holomorphic on U and $f'(z) \neq 0$ for all $z \in U$. Show that

$$\{\mathrm{Re}(f(z)) + \mathrm{Im}(f(z)) : z \in U\} \subset \mathbb{R}$$

is an open subset of $\mathbb{R}$.

(4) Show that there exists a constant $M > 0$ such that for every complex polynomial $p(z)$,

$$\max_{|z|=1} |z^{-1} - p(z)| \geq M.$$

(5) For $0 < a < b$, evaluate

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|ae^{i\theta} - b|^4} d\theta.$$

(6) Let f be holomorphic in the whole complex plane and satisfies

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq r^{17/3}$$

for all $r > 0$. Prove that f is the zero function.

(7) Suppose the function $f = u + iv$ is holomorphic in the whole complex plane and satisfies the inequality

$$u^2 \leq v^2 + 2004$$

in $\mathbb{C}$. Is f constant?

(8) Suppose the function f is holomorphic in the whole complex plane and satisfies the inequality

$$|f(z)| \leq |\mathrm{Re}(z)|^{-1/2}$$

off the imaginary axis. Prove that f is constant.

(9) Suppose the function f is holomorphic in the whole complex plane and for each $z_0 \in \mathbb{C}$, atleast one coefficient c_n in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

(10) Let f be non-constant and holomorphic in a domain containing the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. Show that the image of f contains the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ in each of the following cases:

(a) $|f(z)| = 1$ whenever $|z| = 1$.
(b) $|f(z)| \geq 1$ whenever $|z| = 1$, and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$.