Recall that a function $u : D \subseteq \mathbb{C}$ is a harmonic function on a domain D, if u satisfies the Laplace equation $u_{xx} + u_{yy} = 0$.

Q 1. Is the product of two harmonic functions harmonic? Why, or why not?

Q 2. Give an example of a smooth, real-valued function f of a real variable, and a harmonic function u such that $f \circ u$ is not harmonic. Determine all smooth f such that $f \circ u$ is harmonic for every harmonic u.

Q 3. Show that $ln|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$ but there is no holomorphic function on $\mathbb{C} \setminus \{0\}$ whose real part is the function $ln|z|$.
Hint. Look at the polar co-ordinates version of the Laplace equation.

Q 4. If u is harmonic on a domain D, and $a \in D$, prove the mean-value property

$$u(a) = \frac{1}{2\pi} \int_0^1 u(a + re^{2it\pi})dt$$

where the circle $|z - a| \leq r$ is inside D.
Hint. Use Cauchy's integral formula for $f = u + iv$ which is holomorphic.

Q 5. Let u be harmonic on an open disc D. If u is constant on some non-empty domain $U \subset D$, prove that u must be constant on the whole of D. Can you generalize this to other domains D? Which ones?

Q 6. If u is harmonic on a bounded domain D, attaining its maximum at a point of D, show that u must be constant.

Q 7. If u_1, u_2 are harmonic on a bounded domain D, and are equal on the boundary ∂D, prove that u_1 and u_2 are equal on D.
Hint. Look at $u_1 - u_2$.

Q 8. If u is harmonic on an infinite vertical strip $[s, t] \times (-\infty, \infty)$, and is constant on the vertical lines $Re(z) = s$ and $Re(z) = t$, determine u in the open strip. What about the analogous problem for an open domain between two circles (the circles may not be concentric)?