Q 1. Let D be the open strip $\{z : 0 < Im(z) < \pi\}$. Consider the function $v : \bar{D} \to \mathbb{R}$, defined by

$$v(z) = Im(z/\pi + Cosh(z)).$$

Show that v is harmonic on D and is unbounded. What are its values on the boundary points of D?
Hint for unboundedness: look at values at points on the middle horizontal line of the strip.

Q 2. Consider the principal branch of $z^i = e^{iLog(z)}$. Consider the semicircle C from -1 to 1 lying in the lower half-plane (except the end points) - or the semicircle can also be replaced by any closed curve in the lower half-plane. Prove that $\int_C z^i dz = (1 + e^\pi)(1 - i)/2$.
Hint: Note that $z^{1+i}/(1+i)$ is a primitive for z^i outside $(-\infty, 0]$. The value of the primitive at -1 is interpreted as the limiting value when $z \to -1$ along $Im(z) < 0$.

Q 3. Let f be an entire function satisfying $|f(z)| \leq log(1 + |z|)$ for all z. Prove that f must be a constant function.
Hint: Same idea as in Liouville's theorem, but applied to f' at any point.

Q 4. Show that $\int_C \frac{z^{2024}}{1+2^{2022}+z^{2023}+z^{2024}} dz = -2i\pi$ where C is any circle of radius > 2 centered at the origin.
Hint: Note that all the zeroes of the denominator lie within the interior of C. Then, write the integrand as $a + b/z$+something, where the something is a rational function whose denominator has degree at least two more than that of the numerator.

Q 5. Find $\int_0^\pi \frac{d\theta}{2-\cos\theta}$ using complex integrals. What about $\int_0^\pi \frac{d\theta}{5+4\cos\theta}$ or, more generally, $\int_0^\pi \frac{d\theta}{s+t\cos\theta}$ where a, b are real and $s^2 > t^2$?
Hint: recognize $\frac{d\theta}{2-\cos\theta}$ as $\frac{dz}{f(z)}$, where $z = e^{i\theta}$ and $f(z)$ is a polynomial of degree 2.

Q 6. Firstly, observe that $\int_{C_r} \frac{dz}{z^2 \sin(z)} \to 0$ as $r \to \infty$, where C_r is the boundary of the square $|x|, |y| \leq (2r + 1)\pi/2$. Using this, and the residue theorem, deduce that $\sum_{n\geq 1}(-1)^{n-1}/n^2 = \pi^2/12$.