Complex Analysis - M. Math.
Assignment II
B. Sury
Due by 12th February 2024

Recall that Liouville's theorem, the maximum modulus theorem, definition of conformality, and the open mapping theorem have been covered in Prof. Jaydeb Sarkar's classes. You may need to use some of these results to solve the problems below.

Q 1. Let C be the circle $3e^{it}; 0 \leq t \leq 2\pi$. Suppose z_0 does not lie on C, and suppose

$$f(z_0) = \int_C \frac{2z^2 - z - 2}{z - z_0} dz.$$

Determine $f(2)$.

Q 2. Consider the closed path C traversed once in the anti-clockwise direction on the boundary of the square bounded by the lines $x = \pm 2, y = \pm 2$. Compute:
(a) $\int_C \frac{3z}{2z+1} dz$;
(b) $\int_C \frac{cosh(z)}{z^{2024}} dz$.

Q 3.
(a) Rewrite the function $f(z) = \frac{z}{(z-1)(z-3)}$ as a series of negative and positive powers of $z - 1$ which converges in $0 < |z - 1| < 2$.
(b) Show that $sec(z) = \sum_{n \geq 0} E_{2n} \frac{z^{2n}}{(2n)!}$ has radius of convergence $\pi/2$. Determine a recurrence formula for the Euler numbers E_{2n} that are the coefficients.
(c) Similarly, show that $zcot(z) = \sum_{n \geq 0} b_{2n} \frac{(2z)^{2n}}{(2n)!}$ is a convergent series in the punctured disc $0 < |z| < 2\pi$, and find a recurrence for the 'signed' Bernoulli numbers b_{2n}.

Q 4. Find the largest open disc D around 0 such that the function $f(z) = 2z^2 + z$ maps D injectively to its image. Is f also 1-1 on the points of the boundary of D? Why, or why not?

Q 5. Let f be holomorphic on a domain D of $\mathbb{C}$ such that $f(z)^2 = \overline{f(z)}$ for all $z \in D$. Determine, with proof, all such f.

Q 6. If f is an entire function taking values only inside the upper half-plane H, prove that f must be constant.
Remark. A consequence of the 'Uniformization Theorem' is that the domain $D := \mathbb{C} \setminus \{z_1, z_2\}$ admits H as its holomorphic 'universal covering space'. Using this, one can show 'Picard's little theorem' that a nonconstant entire function cannot miss two points in its image.

Q 7. If f is a nonconstant entire function, then prove that the image of f contains a dense set in $\mathbb{C}$. Using this, or otherwise, prove that if $g = u + iv$ is entire, $M > 0$ is a constant such that $|u(z)| \leq |v(z)|$ for all $|z| \geq M$, then g must be constant.

Q 8. Consider the polynomial $p(z) = az^2 + 2(|a|^2 - 1)z - \bar{a}$, where $|a| \leq 1$ is fixed. Prove that $|p(z)| \leq 1$ if $|z| \leq 1$.