Assignment III : Complex Analysis
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Due by 11th March, 2024.

Q 0. (Rouché - Filaseta) - not for submission.
Try using the following steps to prove it yourself; if not, then consult the article attached here.
Prove that Rouché's theorem for complex polynomials is equivalent to the fundamental theorem of algebra. More precisely, assuming the fundamental theorem of algebra, prove that complex polynomials f, g satisfying $|f(z) + g(z)| < |f(z)| + |g(z)| \ \forall \ z$ on $|z| = 1$, must have the same number of zeroes in the open unit disc $B(0,1)$.
Hint. Observe first that the inequality of the hypothesis implies that neither f nor g have zeroes on the unit circle. If r and s denote the numbers of zeroes of f and g inside $B(0,1)$, then assuming $r > s$, it suffices to get $\theta \in [0, 2\pi]$ with $|f(e^{i\theta}) + g(e^{i\theta})| = |f(e^{i\theta})| + |g(e^{i\theta})|$.

Q 1. (On Rouche' - easy problems).
(a) Let $n \geq 1, a_0 \neq 0$. Then, show that $z^n - |a_{n-1}|z^{n-1} - \ldots - |a_1|z - |a_0|$ has a unique real, positive zero M. Moreover, prove that the polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has all its zeroes in $\bar{B}(0, M)$.
(b) Let $M_0 > 0$ be such that $|a_i| < M_0|a_0| \ \forall \ i = 1, \ldots, n$. Then show that $f = \sum\limits_{i=0}^{n} a_i z^i$ has all its roots in $|z| \geq \frac{1}{M+1}$.

Q 2. (Miscellaneous).
Prove that a polynomial f with complex coefficients has Rolle's property (that is, on the line segment joining two zeroes of f, there is always a zero of f') if, and only if, its zeroes are all collinear.
Hint for necessity. If f has Rolle's property but not all of its zeroes are collinear, let z_1, z_2, z_3 be zeroes of f forming triangle of smallest area among all such triangles. Write $z_4, \ldots, z_r$ for the other *distinct* zeroes of f. Then, none of the $z_4, \cdots, z_r$ can lie on the closed triangle $\overline{\Delta z_1 z_2 z_3}$. Join all the points $z_4, \cdots, z_r$ to one of the vertices z_1, z_2, z_3 in such a way that all the segments are non-intersecting and, each segment intersects $\overline{\Delta z_1 z_2 z_3}$ only in that particular vertex of the triangle to which it is connected. Using the hypothesis that Rolle's property holds for f, observe that there are at least $r - 3$ zeroes of f' which are outside Δ and are different from the zeroes of f. Also, applying Rolle's property for the 3 edges of the triangle, there are 3 more zeroes of f' different from z's. Complete the proof.

Q 3. (On Maximum Modulus.)
For the following domains and functions f, find the maximum of $|f|$ on the closures of these domains:
(a) $f(z) = (z-a)(z-b)(z-c)(z-d)$ on the closure of the square with sides of length ℓ and vertices at a, b, c, d.
(b) $f(z) = z^2 + 2az - 1$ on the closed unit disc, where a is real.

Q 4. (On Schwarz Lemma).
(a) Prove that an analytic function from the open unit disc $B(0,1)$ to itself that has two fixed points, must be the identity; find one with exactly one fixed point.
(b) Let $f : B(0,1) \to B(0,1)$ be analytic, and suppose $f(0) = 0 = f'(0)$. Prove that $|f"(0)| \leq 2$, and that equality holds if, and only if, there is $\lambda \in S^1$ so that $f(z) = \lambda z^2$ for all $z \in B(0,1)$.
(c) Let $f : \mathcal{H} \to D$ be a holomorphic function from the upper half-plane $\mathcal{H}$ to the open unit disc D. If $f(i) = 0$, find the maximum possible value of $|f(2i)|$.

Q 5. (On Rouché's theorem).
(a) Find the number of roots (counted with multiplicity) of $16z^7 + 16z^3 = -1$ in the disc $|z| < 1/2$.
(b) Show that there is exactly one solution of $z + e^{-z} = 2024$, with positive real part.
(c) Show that for any given $R > 0$, there exists n so that the truncated exponential polynomial $\sum_{d=0}^{n} \frac{z^d}{d!}$ has ALL its zeroes in $|z| > R$.
(d) Show that $e^z = cz^n$ has n roots inside the open unit disc, for any real c with $c > e$.
(e)$*$ Show that all the four roots of $z^4 + iz^2 + 2$ lie inside the CLOSED disc $\{z : |z| \leq \sqrt{2}\}$. Further, determine how many lie on the circles of radii 1 and $\sqrt{2}$. Finally, determine the number of roots in each quadrant.

Q 6. (Conformal maps).
Find the image under the Joukowsky map $J : z \mapsto (z + 1/z)/2$, of: (i) $|z| < r$ for $r < 1$, (ii) $|z| > r$ where $r > 1$, (iii) $|z| > 1$, (iv) upper half-plane $Im(z) > 0$, (v) $|z| < 1$ with $Im z > 0$. Show J is conformal, except at ± 1. Argue why there cannot be a conformal isomorphism from the open unit disc to the whole of $\mathbb{C}$.