M. Math. Complex Analysis
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Assignment IV - Mostly Miscellaneous Problems
Due by 29th March 2024

Q 1.
(i) Determine all entire functions f satisfying $f(1-z) + f(z) = 1$ for all $z \in \mathbb{C}$.
(ii) If g is a continuous function on $\mathbb{C}$, and is holomorphic on the complement of the segment $[-1, 1]$, then show that g is entire.

Q 2. If f is a holomorphic function in a domain D and $a \in D$, show that we cannot have the property $|f^{(n)}(a)| \geq n^n n!$ for all $n \geq 1$.

Q 3.
(i) Prove that $e^{-z} + z = 3$ has a unique root on the right half-plane $\Re(z) > 0$.
(ii) Show that the equation $1 + z + z^4$ has exactly one root in each quadrant, and that all of them lie in $B(0, 3/2)$.
(iii) Show that $\{z \in \mathbb{C} : z\sin(z) = 1\} \subset \mathbb{R}$.

Q 4.
(i) If f is a **bounded** holomorphic function on $B(0, 1)$ and has zeroes at $\{\zeta_n\}_n$, then prove $|f'(0)| \leq \prod_{r=1}^{n} |\zeta_r|$ for each $n \geq 1$.
(ii) If f is as in (i), and is not the zero function, then prove that $\sum_n \log|\zeta_n|$ converges.
(iii) Let g be holomorphic on the right half-plane $\Re(z) > 0$, and if $|g(z)| \leq M$ for all $\Re(z) > 0$, and if $\{w_n\}$ are the zeroes of g, then $|g(z)| \leq \prod_{r=1}^{n} \left|\frac{z - w_r}{z + \overline{w_r}}\right|$ for all $\Re(z) > 0$, and for all $n \geq 1$.
(iv) Finally, with g as above which is not the zero function, prove that $\sum_n \Re(1/w_n)$ converges.

Q 5. Prove that there is no conformal isomorphism between the sets $\mathbb{C} \setminus \{0, 1, 2\}$ and $\mathbb{C} \setminus \{0, 1, 2024\}$.

Q 6. Mark off points on the unit circle, dividing the circumference into n equal parts. Fix one of these points and, moving clockwise along the circumference, join this point to the k-th points for each k coprime to n. Show that the products of the lengths of these chords equals p if n is the power of a prime p, and equals 1 otherwise.

Q 7. (Waring problem for polynomials).
Let $n \geq 3$. If $a(n)$ denotes the smallest positive integer such that X is a sum of n-th powers of $a(n)$ complex polynomials, then prove that $a(n) \geq 3$ and $a(n) \leq n < a(n)^2 - a(n)$.

Q 8. Write out in detail the proof of the following theorem of Bernstein on polynomials using two different methods outlined below:
(Bernstein) If f is a polynomial, then $\|f'\| \leq deg(f)\|f\|$, where $\|f\| := max\{|f(z)| : |z| = 1\}$.
Proof 1: We shall use the Gauss-Lucas theorem (that zeroes of the derivative of a polynomial lie in the convex hull of the zeroes of the polynomial) and the Rouché's theorem, we can give a quicker proof. Let g be a polynomial of the same degree, say n, as f and with no zeroes with $|z| \geq 1$. We show that if $|f(z)| < |g(z)|$ on the unit circle, then $|f'(z)| < |g'(z)|$ for $|z| \geq 1$. Apply Rouché's theorem to the polynomial $f(z) + tg(z)$ for any $|t| \geq 1$ to deduce that all its zeroes lie in the open unit disc. By the Gauss-Lucas theorem, the same is true for the polynomials $f'(z) + tg'(z)$ for every t with $|t| \geq 1$. If $|f'(z_0)| \geq |g'(z_0)|$ for some point z_0 with $|z_0| \geq 1$, then we would be able to choose some t with $|t| \geq 1$ such that $f'(z) + tg'(z)$ has a zero in $|z| \geq 1$, thereby contradicting what we observed earlier. Therefore, we must have $|f'(z)| < |g'(z)|$ for all $|z| \geq 1$. So, if we consider $g(z) = cz^n$ where $c > max\{|f(z)| : |z| = 1\}$, the above can be applied to this g and we get $|f'(z)| < |g'(z)| = nc$ for $|z| \geq 1$. This proves Bernstein's inequality.
Proof 2: Let $deg(f) \leq n$. Consider the roots $z_1, \cdots, z_n$ of the polynomial $X^n + 1$; these are the roots of unity $e^{\pi ri/n}$ with r odd and $\leq 2n - 1$. Then, for any $z_0 \in \mathbb{C}$, the polynomial $g_0(z) := \frac{f(z_0 z) - f(z_0)}{z-1}$ has degree $\leq n - 1$ and satisfies $g_0(1) = z_0 f'(z_0)$. Interpolating at the z_i's $(1 \leq i \leq n)$, we have

$$g_0(z) = \sum_{r=1}^{n} \frac{z^n + 1}{(z - z_r)nz_r^{n-1}} g_0(z_r) = \frac{1}{n} \sum_{r=1}^{n} \frac{z^n + 1}{z_r - z} z_r g_0(z_r).$$

Evaluating at $z = 1$, we have

$$z_0 f'(z_0) = g_0(1) = \frac{1}{n} \sum_{r=1}^{n} \frac{2z_r g_0(z_r)}{z_r - 1} = \frac{1}{n} \sum_{r=1}^{n} \frac{2z_r \Big(f(z_0 z_r) - f(z_0) \Big)}{(z_r - 1)^2}.$$

This can be simplified by proving that

$$\sum_{r=1}^{n} \frac{2z_r}{(z_r - 1)^2} = \frac{-n^2}{2}.$$

Deduce

$$z_0 f'(z_0) = \frac{n}{2} f(z_0) + \frac{1}{n} \sum_{r=1}^{n} \frac{2 z_r f(z_0 z_r)}{(z_r - 1)^2}.$$

Considering z_0 on the unit disc, we have

$$|f'(z_0)| \leq \frac{n}{2} \|f\| + \left(\frac{1}{n} \sum_{r=1}^{n} \left| \frac{2 z_r}{(z_r - 1)^2} \right| \right) \|f\|.$$

Observe that each $\frac{2z_r}{(z_r-1)^2}$ is a real number of the same sign, and conclude that

$$\frac{1}{n} \sum_{r=1}^{n} \left| \frac{2 z_r}{(z_r - 1)^2} \right| = \frac{n}{2}.$$

Therefore, $|f'(z_0)| \leq n\|f\|$ for all z_0 on the unit circle.