Riemann Surfaces - some basic problems

Q 1. Let $\psi : \mathbb{C} \to \mathbb{C}$ be given by complex conjugation $\psi(z) = \bar{z}$. Show that $(\mathbb{C}; \psi)$ defines a Riemann surface structure (denoted by $\overline{\mathbb{C}}$) on $\mathbb{C}$ which has a different maximal atlas from the standard Riemann surface structure on $\mathbb{C}$. Show that $\mathbb{C}$ and $\overline{\mathbb{C}}$ are biholomorphic as Riemann surfaces.

Q 2. Suppose X is a Riemann surface and $\phi : D^* \to X$ is a holomorphic map. Define

$$X \cup_\phi D := (X \cup D)/\sim$$

where the equivalence relation identifies $z \in D^*$ with $\phi(z) \in X$. Consider the atlas of $X \cup_\phi D$ which consists of all of the charts of X together with one additional chart given by the inverse of the obvious map from D to $X \cup_\phi D$. Prove that $X \cup_\phi D$ is a Riemann surface if and only if ϕ does not extend to a holomorphic map from D to X.
Hint: Show that $X \cup_\phi D$ is Hausdorff if and only if ϕ does not extend to a holomorphic map from D to X.

Q 3. Let $P(z_1; z_2)$ be a polynomial and $\hat{P}(z_0; z_1; z_2)$ be its homogenization. Write $X_P = \{(z_1, z_2) \in \mathbb{C}^2 : P(z_1, z_2) = 0\}$ and $\hat{X}_{\hat{P}} = \{[z_0; z_1; z_2] : \hat{P}(z_0, z_1, z_2) = 0\}$.
(a) Show that $\hat{X}_{\hat{P}} \setminus X_P$ is a finite set of points and give an upper bound for the number of points in terms of some property of P.
(b) Consider $P(z_1, z_2) = z_1^k - z_2$. In homogeneous coordinates $[z_0; z_1; z_2]$ compute $\hat{X}_{\hat{P}} \setminus X_P$.
(c) What is the topological surface for the Riemann surface $\hat{X}_{\hat{P}}$ when $k = 2$?

Q 4. Let $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ for $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ not real multiples of each other.
(a) Explicitly prove that $\mathbb{C}/\Lambda$ is a Riemann surface (i.e. construct charts).
(b) Prove that $\mathbb{C}/\Lambda$ is equivalent to $\mathbb{C}/\Lambda'$ where $\lambda' = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in H$. What is the relation between τ and ω_1, ω_2?
(c) Show that $z \mapsto -z$ defines an automorphism $i \in Aut(\mathbb{C}/\Lambda)$ with $i \neq Id$ but $i^2 = Id$.
(d) How many fixed points does i have? That is, how many $w \in \mathbb{C}/\Lambda$ have the property that $i(w) = w$?
(e) Prove that translation $z \mapsto z+b$ defines an automorphism $T_b \in Aut(\mathbb{C}/\Lambda)$. Interpret this as defining a group homomorphism $\mathbb{C} \to Aut(\mathbb{C}/\Lambda)$ and compute the kernel.

(f) (bonus) Find two lattices Λ such that $\mathbb{C}/\Lambda$ has an automorphism which is not a translation or a translation composed with i. It turns out there are only two such lattices.
(g) What is the topology of the quotient $(\mathbb{C}/\Lambda)/(\mathbb{Z}/2)$ where $\mathbb{Z}/2$ acts by i?