Harmonic Analysis - B. Math. III

Worksheet — 2nd Semester 2023-2024 (Final)

1 Haar Integration

1. Show that the multiplication mapping $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (given by $(x, y) \mapsto xy$) is not a closed map.

2. Let G be a topological group and suppose there exists a compact subset K of G such that $xK \cap K \neq \varnothing$ for every $x \in G$. Show that G is compact.

3. Let G be a locally compact group with Haar measure μ, and let $S \subseteq G$ be a measurable subset with $0 < \mu(S) < \infty$. Show that the map $x \mapsto \mu(S \cap xS)$ from G to R is continuous.

4. Let B be the subgroup of $GL_2(\mathbb{R})$ defined by

$$B = \left\{ \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R}, y \neq 0 \right\}.$$

   Show that $I(f) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} f \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} dx \frac{dy}{y}$ is a Haar-integral on B. Show that the modular function Δ of B satisfies:

$$\Delta \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} = |y|.$$

5. Let $\mathbb{C}^\times$ be the multiplicative group of non-zero complex numbers. Let $\mathbb{R}_{>0}$ denote the multiplicative group of positive real numbers. Show that $\mathbb{C}^\times$ is isomorphic to $\mathbb{R}_{>0} \times \mathbb{T}$ (where $\mathbb{T}$ is the circle group) under the polar decomposition map $(r, u) \mapsto ru$. We write $u = e^{2\pi i\theta}$. Show that the Haar integral on $\mathbb{C}^\times$ is given by

$$f \mapsto \int_0^1 \int_0^\infty f(re^{2\pi i\theta}) \, \frac{dr}{r} \, d\theta.$$

6. Let $G = GL_n(\mathbb{R})$ be the group of real $n \times n$ matrices. Show that Haar measure on G is given by $dx/|\det x|$, if dx is a Haar measure on the n^2-dimensional space of all $n \times n$ matrices.

7. If Δ is the modular function on G, show that

$$\int_G f(x^{-1})\Delta(x^{-1}) \, dx = \int_G f(x) \, dx,$$

   where dx is a Haar measure on G. Show that $\Delta(x^{-1})dx$ is a right Haar measure.

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8. Compute the modular function for the group G of all affine maps $x \mapsto ax+b$ with $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$. (In this case, the right Haar measure is not equal to the left Haar measure.) Show that the right Haar measure is the Cartesian product measure on $\mathbb{R}^\times \times \mathbb{R}$.

9. Let G, H be locally compact groups and assume that G acts on H (via $\pi : G \to Aut(H)$) by group homomorphisms $h \mapsto \pi(g)h$, such that the ensuing map $G \times H \to H$ is continuous.

   (i) Show that the product $(h, g)(h', g') = (h \cdot \pi(g)h', gg')$ gives $H \times G$ (with the product topology) the structure of a locally compact group, called the semi-direct product $H \rtimes G$.

   (ii) Show that there is a unique group homomorphism $\delta : G \to (0, \infty)$ such that $\mu_H(\pi(g)A) = \delta(g)\mu_H(A)$, where μ_H is a Haar measure on H and A is a measurable subset of H.

   (iii) Show that $\int_H f(\pi(g)x)d\mu_H(x) = \delta(g) \int_H f(x)d\mu_H(x)$ for $f \in C_c(H)$ and deduce that δ is continuous.

   (iv) Show that a Haar integral on $H \rtimes G$ is given by

   $$\int_H \int_G f(h, g)\delta(g)d\mu_H(h)d\mu_G(g).$$

   (v) What is the right Haar measure of $H \rtimes G$?

2 Banach algebras

10. Let A be a complex Banach algebra with unit element, and let $u \in A$. Let $\sigma_A(u)$ be the spectrum of u. Let p be a polynomial with complex coefficients. Show that $\sigma_A\big(p(u)\big)$ is equal to $p(\sigma_A(u)) := \{p(\alpha) : \alpha \in \sigma(u)\}$.

11. Let A be a unital Banach algebra and $x, y \in A$. Prove that $xy - yx \neq 1$. In other words, the Heisenberg commutation relation cannot be realized in Banach algebras. (Hint: Show that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.)

12. Give an example of a unital Banach algebra A and two elements $x, y \in A$ with $xy = 1$, but $yx \neq 1$.

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13. Let $(A, \|\cdot\|)$ be a Banach algebra. Show that for every $a \in A$, the series

$$\exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

converges and that, for $a, b \in A$ with $ab = ba$, one has $\exp(a+b) = \exp(a)\exp(b)$.

14. Let $A = C(X)$ for a compact Hausdorff space X. For $x \in X$ let $m_x : A \to \mathbb{C}$ be defined by $m_x(f) = f(x)$. Show that the map $x \mapsto m_x$ is a homeomorphism from X to the structure space Δ_A.

15. (Wiener's Lemma) Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function such that

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \text{ with } \sum_{n\in\mathbb{Z}} |a_n| < \infty.$$

Show that if $f(x) \neq 0$ for every $x \in \mathbb{R}$, then there exist $b_n \in \mathbb{C}$ such that

$$\frac{1}{f(x)} = \sum_{n\in\mathbb{Z}} b_n e^{inx} \text{ with } \sum_{n\in\mathbb{Z}} |b_n| < \infty.$$

16. Let A and B be commutative C^*-algebras, and let $\phi : A \to B$ be a linear map with $\phi(aa') = \phi(a)\phi(a')$ for any $a, a' \in A$. Show that ϕ is a continuous $*$-homomorphism.

3 Duality for LCA groups

Let G be a locally compact group. The *character group, (Pontryagin) dual group* of G is the (abelian) group $\widehat{G}$ of characters of G, under the pointwise product, equipped with the compact-open topology.

17. Let G be a locally compact group. Show that $\widehat{G}$ is an LCA group, and that the assignment $G \mapsto \widehat{G}$ is a contravariant functor, in the sense that for any continuous homomorphism $\varphi : G' \to G$, there is a dual continuous homomorphism $\widehat{\varphi} : \widehat{G} \to \widehat{G'}$ given by $\widehat{\varphi}(\chi) = \chi \circ \varphi$.

18. A topological space is called *second countable* if its topology admits a countable base. Show that if an LCA-group A is second countable, then so is its dual $\widehat{A}$.

19. Let A and B be two LCA groups. Show that $\widehat{A \times B} \cong \widehat{A} \times \widehat{B}$.

20. Show that the multiplicative group $\mathbb{C}^\times$ is locally compact with the topology of $\mathbb{C}$ and that $\widehat{\mathbb{C}^\times} \cong \mathbb{Z} \times \mathbb{R}$.

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21. Let A be an LCA group. Prove that $C^*(A) \cong C_0(\widehat{A})$ as C^*-algebras.

22. Let A be an LCA group, and let $f \in L^1(A)$. Show that $\widehat{f} \in C_0(\widehat{A})$. Using the previous exercise, show that the Fourier transform $L^1(A) \to C_0(\widehat{A})$ is injective.

23. Let A be an LCA group, and let $f \in L^1(A)$ such that $f \in L^1(\widehat{A})$. Show that $f \in L^2(A)$.

24. Let A be an LCA group, and consider the mapping $x \mapsto \delta_x$ from A to $\widehat{\widehat{A}}$ where $\delta_x(\chi) = \chi(x)$. Prove that the mapping is a homeomorphism when A is isomorphic to $K \times \mathbb{R}^n \times \mathbb{Z}^m$ where K is a compact abelian group and $m, n \geq 0$. (These are the compactly generated LCA groups. After proving that every LCA group is a union of its open compactly generated subgroups and noting compatibility of Pontryagin duals with limits, this provides another derivation of Pontryagin duality.)

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