Harmonic Analysis - B. Math. III

Worksheet — 2nd Semester 2023-2024

1 Fourier Analysis

1. Let C_n be the cyclic group of order n.

   (i) Prove that $\mathbb{C}[C_n]$ is isomorphic to the direct sum of n copies of the field $\mathbb{C}$.

   (ii) Is the analogous assertion true for the field $\mathbb{R}$?

2. Let G be a finite group and $\mathbb{K}$ be a field. Suppose that the algebra $\mathbb{K}[G]$ is commutative. Does it follow that G is commutative?

3. Explicitly compute the convolution of the indicator functions $\chi_{[a,b]}$ and $\chi_{[c,d]}$ viewed as functions on $\mathbb{R}$.

4. Prove that the convolution of two bounded functions in $L^1(\mathbb{R}, dx)$ is a continuous function.

5. Compute the convolution $f_1 * f_2$ in $L^1(\mathbb{R}, dx)$ if:

   (i) $f_1(x) = \frac{1}{x^2+a^2}, f_2(x) = \frac{1}{x^2+b^2}$;

   (ii) $f_1(x) = e^{-x^2/2a}, f_2(x) = e^{-x^2/2b}$.

6. (i) Suppose that the function φ on $\mathbb{R}$ coincides inside the interval $[-N, N]$ with some polynomial and equals zero outside this ball, and that the function $\psi \in L^1(\mathbb{R}, dx)$ has support in the interval $[-n, n]$ for $n < N$. Prove that the convolution $\varphi * \psi$ has support in $[-N-n, N+n]$ and coincides with some polynomial in the ball of radius $[-(N-n), N-n]$.

   (ii) (Weierstrass approximation theorem) Prove that every continuous function on $\mathbb{R}$ can be approximated uniformly by polynomials on any compact set.

7. Find the explicit form of the characters on the cyclic group C_n of order n.

8. Prove that every finite abelian group is isomorphic (not canonically) to its dual group $\hat{G}$.

9. A *generalized character* on a group G is defined to be a continuous homomorphism of of G into the multiplicative group of $\mathbb{C}$ (denoted by $\mathbb{C}^\times$).

   (i) Prove that for a compact group G, the generalized characters are the usual unitary characters.

   (ii) Find the generalized characters and unitary characters of $\mathbb{Z}^n, \mathbb{R}^n, \mathbb{C}^n, \mathbb{R}^\times, \mathbb{C}^\times$.

Soumyashant Nayak: soumyashant@isibang.ac.in

2 Distributions

10. (i) Prove that there is no function δ in $C_c(\mathbb{R})$ such that $\delta * f = f$ for all $f \in C_c(\mathbb{R})$.

    (ii) Prove that there is no function δ in $L^1(\mathbb{R})$ such that $\delta * f = f$ for all $f \in L^1(\mathbb{R})$.

11. Show that the following applications are distributions.

    (i) $\varphi \in C_c^\infty(\mathbb{R})^\# \mapsto \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx$. (This distribution is called the principal value of $\frac{1}{x}$, and denoted by $pv(\frac{1}{x})$.)

    (ii) $\varphi \in C_c^\infty(\mathbb{R})^\# \mapsto \lim_{\varepsilon \to 0} \left(\int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x^2}\, dx \right) - \frac{\varphi(0)}{\varepsilon} + \varphi'(0) \log \varepsilon$.

12. Let δ_0 be the Dirac delta distribution supported at $\{0\}$. Prove that

$$x^m \frac{d^n \delta_0}{dx^n} = \begin{cases} 0 & \text{if } m > n \\ \frac{(-1)^m n!}{(n-m)!} \frac{d^{n-m}\delta_0}{dx^{n-m}} & \text{if } m \leq n. \end{cases}$$

13. Prove that

$$\frac{d}{dx} \log |x| = pv(\frac{1}{x})$$

    in the sense of distributions.

14. For $\varepsilon > 0$, set $T_\varepsilon = \frac{\varepsilon}{2} |x|^{\varepsilon - 1}$. Prove that $\lim_{\varepsilon \to 0} T_\varepsilon$ exists in the sense of distributions. What is the limiting distribution?

15. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers such that

$$|a_n| \leq C n^p, \text{ for some } p \in \mathbb{N}.$$

    (i) Prove that the sequence $S_N = \sum_{n=-N}^{N} a_n e^{2\pi i n x}$ converges in the sense of distributions. We shall denote its limit by $S = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$.

    (ii) Prove that

$$\frac{dS}{dx} = \sum_{n \in \mathbb{Z}} (2\pi i n) a_n e^{2\pi i n x}$$

    and that $\tau_1(S) = S$ where τ_1 denotes right-translation by 1. (Guess the definition of translations for a distribution first.)

    (iii) Prove that the Fourier transform of δ_n is the function $e^{-2\pi i n x}$.

    (iv) Let $T = \sum_{n \in \mathbb{Z}} e^{2\pi i n x}$. Prove that $(1 - e^{2\pi i n x})T = 0$ and deduce that $T = \sum_{n \in \mathbb{Z}} \delta_n$. In other words,

$$\sum_{n \in \mathbb{Z}} e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} \delta_n.$$

    (This should remind you of the Poisson summation formula.)

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3 Fourier Transform

16. Prove that the Fourier transform gives a homeomorphism of the Schwartz space, $\mathcal{S}(\mathbb{R})$.

17. Prove that the family of Schwartz functions $\{e^{-(x-a)^2)}\}_{a\in\mathbb{R}}$ spans a dense subspace of $\mathcal{S}(\mathbb{R})$.

18. Let $D = \frac{d}{dx}$, and M denote the operator of multiplication by x. Define the operators $A := D+M, A^* := D-M$ (the so-called *creation* and *annihiliation* operators in quantum field theory) acting on the Schwartz space, $\mathcal{S}(\mathbb{R})$.

    (i) Prove that the equation $Af = 0$ has a one-dimensional space of solutions in $\mathcal{S}(\mathbb{R})$. (Hint: $e^{-x^2/2}$ is a solution.)

    (ii) Let f_0 be a basis vector in the solution space of the system $Af = 0$ (the so-called *vacuum vector*). We assume $f_0 = e^{-x^2/2}$ for the rest of this exercise. Let $f_m = (A^*)^m f_0$. Using induction, show that $f_m = e^{x^2/2}D^m(e^{-x^2})$.

    (iii) Use Taylor series expansion to show that

    $$e^{-(x+a)^2} = e^{-x^2/2}\left(\sum_{m=0}^{\infty}\frac{a^m}{m!}f_m\right), \forall a \in \mathbb{R}.$$

    (iv) Prove that the system of functions $(f_m)_{m\in\mathbb{N}_0}$, spans a dense subset of $\mathcal{S}(\mathbb{R})$.

    (iv) Show that $DM - MD = I$ on $\mathcal{S}(\mathbb{R})$. Using this, note that $\frac{1}{2}(AA^* - A^*A) = I$.

    (v) Let $N = \frac{1}{2}A^*A$ (so-called *occupation number operator*). Prove that the functions $f_m, m \in \mathbb{N}$, are eigenfunctions for the operator N, and compute the corresponding eigenvalues.

    (v) Prove that the mapping $\varphi \mapsto (\langle\varphi, f_m\rangle_2)_{m\in\mathbb{N}_0}$ gives a bijection between $\mathcal{S}(\mathbb{R})$ and the space of sequences $(c_m)_{m\in\mathbb{N}_0}$ with the property that $|c_m| = o(|m|^{-k})$ for all $k \in \mathbb{N}_0$.

    (vi) Compute the Fourier transforms of the functions, f_m.

19. Let P be a polynomial on $\mathbb{R}$ of degree $2m$ without any real roots.

    (i) Prove that the Fourier transform of the function $f(x) = \frac{1}{P(x)}$ is infinitely differentiable everywhere except at 0.

    (ii) Prove that $\hat{f}$ has one-sided derivatives of all orders at 0.

20. Suppose that $f \in \mathcal{S}(\mathbb{R})$ and $\int_{\mathbb{R}} x^n f(x)\, dx = 0$ for all $n \in \mathbb{N}$. Does it follow that $f \equiv 0$?

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