HOMEWORK 3 (due APRIL 9)

1. Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function. Show that for all $y_1 \in \partial\varphi(x_1)$, $y_2 \in \partial\varphi(x_2)$

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0$$

2. For any proper (not identically $+\infty$) function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ one can define its convex conjugate function / Legendre transform, by

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} \left(x \cdot y - \varphi(x) \right)$$

a) Show that φ^* is proper, convex and lower semi-continuous, i.e.

$$\liminf_{k \to \infty} \varphi^*(x_k) \geq \varphi^*(x) \quad \text{for every } x_k \to x$$

b) Show $\forall\, x, y \in \mathbb{R}^n$

$$x \cdot y \leq \varphi(x) + \varphi^*(y)$$

3. Let φ be a proper lower semi-continuous convex function on $\mathbb{R}^n$. Show for all $x, y \in \mathbb{R}^n$

$$x \cdot y = \varphi(x) + \varphi^*(y) \iff y \in \partial\varphi(x)$$