REPRESENTATION THEORY OF FINITE GROUPS
ASSIGNMENT-IA

(1) Show that a representation $\phi : G \to \mathrm{GL}(V)$ is irreducible if and only if for all nonzero $v \in V$, the set $\{\phi_g(v); g \in G\}$ spans V.

(2) Let S_n denote the symmetric group on n elements. Let $\rho : S_n \to \mathrm{GL}(\mathbb{C}^n)$ be the permutation representation, ie.

$$\rho_\sigma(e_i) = e_{\sigma(i)},$$

for all $\sigma \in S_n$, and $e_1, \ldots, e_n$ denoting the standard basis of $\mathbb{C}^n$.

(a) Show that the following are subrepresentations:

$$V_1 = \{\sum_{i=1}^{n} a_i e_i : a_1 = a_2 = \ldots = a_n\}$$

and

$$V_2 = {V_1}^{\perp} = \{\sum_{i=1}^{n} a_i e_i : \sum a_i = 0\}.$$

(Note that V_2 is the standard representation of S_n.)

(b) Determine the dimensions of V_1 and V_2.

(c) Show that V_1 and V_2 are irreducible representations.

(d) Show that these are the only nonzero proper subrepresentations of ρ.

(3) Let G be a group with commutator subgroup G'. Then prove the following:

(a) $G' = \cap \ker\phi$, where the intersection runs over all one-dimensional representations ϕ of G.

(b) Index of G' in G is the number of one-dimensional representations of G. (c) Find the number of one-dimensional representations of S_n, $n \geq 2$.

(d) Find the number of one-dimensional representations of A_n, for $n \geq 2$.

(4) Let G is a non-abelian group of order p^3, for some prime p.

(a) Show that the order of the commutator subgroup G' is p.

(b) Show that $G/G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, ie. G/G' is abelian but not cyclic.

(c) Construct p^2 distinct irreducible 1-dimensional representations of $\mathbb{Z}_p \times \mathbb{Z}_p$ and justify these are all the irreducible representations of $\mathbb{Z}_p \times \mathbb{Z}_p$.

(d) Show that G has p^2 irreducible representations of dimension 1 and $p - 1$ irreducible representations of dimension p.

(e) Conclude that G has $p^2 + p - 1$ conjugacy classes.

(5) Show that a group G is abelian if and only if all irreducible representations are 1-dimensional.

(6) Find all the representations for the dihedral group D_{2n} of order $2n$,

(a) for n even,

(b) for n odd.

(7) Consider the representations of S_4 and their restrictions to A_4.

(a) Which are still irreducible when restricted to A_4, and which are no more irreducible?

(b) Which pairs of nonisomorphic representations of S_4 become isomorphic when restricted?

(c) Which representations of A_4 arise as restrictions from S_4?

(8) Write down the character table for the group of quaternions Q_8.

(9) Write down the character table for the group S_5 and A_5 with complete justifications.

(10) Let $C_4 = \{1, r, r^2, r^3\}$ denote the cyclic subgroup of order 4 of the dihedral group $D_8 = \langle r, s : r^4 = e, s^2 = e, rs = sr^3 \rangle$. Note that C_4 has an irreducible representation $\phi : C_4 \to \mathbb{C}^\times$ such that $\phi_r = e^{\pi i/2}$. Show by Frobenius reciprocity that the induced representation $\mathrm{Ind}_H^G \phi$ of D_8 is also irreducible.