REPRESENTATION THEORY- PRACTICE SHEET

Note: G is a finite group and all representations are complex vector spaces.

(1) Let (V, ϕ) and (W, ρ) be two G-representations. Show that $V^* \otimes W$ and $\text{Hom}(V, W)$ are equivalent representations of G.

(2) Show that if V is irreducible if and only if V^* is also irreducible.

(3) Let $H = \{e, r, r^2, r^3\}$ be the subgroup of $D_8 = \langle r, s / r^4 = e, s^2 = e\rangle$. Let $\phi : H \to \mathbb{C}^*$ be the 1-dimensional representation of H such that $\phi(r^k) = i^k$. Find the decomposition of the induced representation $\text{Ind}_H^{D_8}\phi$ into irreducible representations of D_8.

(4) Let ϕ be the standard representation of S_3. Decompose the induced representation $\text{Ind}_{S_3}^{S_4}\phi$ into irreducibles of S_4.

(5) Let $\rho : G \to GL(V)$ be a representation of degree n. Define a subset of G by

$$\ker\chi_\rho = \{g \in G : \ \chi_\rho(g) = n\}.$$

Show that $\ker\chi_\rho = \ker\rho$, and hence $\ker\chi_\rho$ is a normal subgroup of G.

(6) Show that $\cap_{k=1}^{r}\ker\ \chi_k = \{e\}$, where $\chi_1, \ldots, \chi_r$ are the distinct irreducible characters of G.

(7) Let $\chi_1, \ldots, \chi_r$ be the distinct irreducible characters of G, and let $X = \{1, \ldots, r\}$. Then a subgroup N of G is normal if and only if it has the form

$$N = \cap_{k\in S}\ker\chi_k$$

for some subset $S \subseteq X$.

(8) G is simple if and only if for every irreducible non-trivial character χ_k, for $g \neq e$, $\chi_k(g) \neq \chi_k(e)$. Hence the character table can be used to decide whether G is simple.

(9) Show that if H is an abelian subgroup of a group G, and d is the degree of any irreducible representation of G, then d is at most the index of H in G.

(10) Show that every irreducible representations of D_n are of degree 1 or 2.