Stochastic Processes: Assignment 1

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Submit solutions to Q.1, Q.5, Q.6 and Q.9 by January 31, 11 PM.

1. If $M_X(s) < \infty$ for all $|s| < s_0$, then show that for all $|s| < s_0$, $\frac{d^k}{ds}M_X(s) = \mathbb{E}\{X^k e^{sX}\}$ and the following series expansion holds:
$$M_X(s) = \sum_{k \geq 0} \mathbb{E}\{X^k\} \frac{s^k}{k!}.$$

2. Consider the ER random graph. Let P be a monotonic increasing graph property. Show that $\mathbb{P}\{G(n,p_1) \in P\} \leq \mathbb{P}\{G(n,p_2) \in P\}$ for $p_1 \leq p_2$.

3. Let $A_1, \ldots, A_n$ be events that $A = \cup_i A_i$. Define
$$S^{(r)} := \sum_{1 \leq i_1 < \ldots < i_r \leq n} \mathbb{P}\{A_{i_1} \cap \ldots A_{i_r}\} \;; X_n = \sum_{i=1}^{n} A_i.$$
Assume that there exists $\lambda > 0$ such that for all $r \in \mathbb{N}$,
$$S^{(r)} \to \frac{\lambda^r}{r!}.$$
Show that $\mathbb{P}\{X_n = 0\} \to e^{-\lambda}$.

4. Consider the ER random graph with $p = \lambda/n$ for $\lambda > 0$. Let D_n be the degree of vertex 1 in $G(n,p)$ and P_λ be the Poisson distribution with mean λ. Show that $\mathbb{E}\left\{D_n^{(r)}\right\} \stackrel{n\to\infty}{\to} \mathbb{E}\left\{P_\lambda^{(r)}\right\}$ for all $r \in \mathbb{N}$ where $k^{(r)} = k!/(k-r)!$ for $k \geq r$ and $k^{(r)} = 0$ for $k < r$.

5. Consider the ER random graph with $np = \log n + c$, $c \in \mathbb{R}$ and n large such that $p_n \in [0,1]$. Let $I(n,p)$ be the number of isolated vertices in $G(n,p)$. Set $\lambda = e^{-c}$. Show that $\mathbb{E}\{I(n,p)^{(r)}\} \stackrel{n\to\infty}{\to} \mathbb{E}\left\{P_\lambda^{(r)}\right\}$ for all $r \in \mathbb{N}$

6. Prove the Chernoff-Cramer bounds for Poisson random variable as in Theorem 2.4.7 of Roch.

7. Let $G = (V,E)$ be a locally-finite infinite graph on a countable vertex set V and let $N(v,n)$ be the number of self-avoiding paths (i.e., non-intersecting paths) of length n starting from $v \in V$. We define the *connective constant* of the graph as
$$\kappa := \sup_{v \in V} \limsup_{n \to \infty} N(v,n)^{1/n}.$$
Show that if G has maximal degree Δ, then $\kappa \leq \Delta - 1$. If $G = \mathbb{Z}^d$, show that $d \leq \kappa \leq 2d - 1$.

8. Let G be a graph as above with finite connective constant $\kappa < \infty$. Let $G(p)$ be the random graph obtained by deleting edges in G independently with probability p. Show that for $p < \kappa^{-1}$,

$$\mathbb{P}\{\text{there exists an infinite path in } G(p) \text{ from some vertex v}\} = 0.$$

9. Let $X_1, \ldots, X_n, \ldots$ be sub-Gaussian random variables (not necessarily independent or identically distributed) with mean μ and variance factor σ^2. Show that a.s.

$$\limsup_{n\to\infty} \frac{\max\{X_1, \ldots, X_n\} - \mu}{\sqrt{\log n}} \le 2\sigma^2.$$

Can you give an example of a sequence where the RHS is σ^2.

10. Let $X_1, \ldots, X_n, \ldots$ be Exponential random variables (not necessarily independent) with mean 1. Show that a.s.

$$\limsup_{n\to\infty} \frac{\max\{X_1, \ldots, X_n\}}{\log n} \le 2.$$

Can you give an example of a sequence where the RHS is 1.

11. Let $X_{i,t}, i, t \ge 1$ be i.i.d. $\mathbb{Z}_+$-valued random variables with finite mean μ and pmf p_k. Suppose that $p_i < 1$ for all $i \ge 0$ and $p_0 > 0$. Define the process Z_t as follows: $Z_0 = 1, Z_k = \sum_{i=1}^{Z_{k-1}} X_{i,k}$. Show that $\mathbb{P}\{Z_k = 0\} \to 1$ if $\mu < 1$.