# Stochastic Processes: Assignment 2

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**There will be a Quiz from these problems on Feb 15th (Thursday) from 9-11 AM.**

1. Let $\mathcal{X} = \mathcal{X}_1 \times \ldots \mathcal{X}_n$ and $f : \mathcal{X} \to mR$ be a function. Let $X_1, \ldots, X_n$ be independent random variables, $Y = f(X_1, \ldots, X_n)$. Show that

$$\mathsf{VAR}(Y) \leq \sum_{i=1}^{n} \mathbb{E}\{(Y - Y^{(i)})_+^2\} = \sum_{i=1}^{n} \mathbb{E}\{(Y - Y^{(i)})_-^2\} \leq 2\sum_{i=1}^{n} \mathbb{E}\Big\{(Y - f_i(X_1, \ldots, \hat{X}_i, \ldots, X_n))^2\Big\},$$

   where $f_i : \mathcal{X}_1 \times \ldots \hat{\mathcal{X}}_i \times \ldots \mathcal{X}_n \to \mathbb{R}$, $1 \leq i \leq n$ are functions.

2. Let $X$ be a mean zero random variable. Show that the following are equivalent.

   (a) $\mathbb{P}\{|X| \geq t\} \leq 2e^{-c_1 t^2}$, $t \geq 0$.

   (b) $\mathbb{E}\{|X|^p\}^{1/p} \leq c_2\sqrt{p}$ for all $p \geq 1$.

   (c) $\mathbb{E}\Big\{e^{\lambda^2 X^2}\Big\} \leq \exp\{c_3^2\lambda^2\}$ for all $|\lambda| \leq c_3^{-1}$.

   (d) $\mathbb{E}\Big\{e^{c_4^2 X^2}\Big\} \leq 2$ for some $c_4$.

   (e) $\mathbb{E}\Big\{e^{\lambda X^2}\Big\} \leq \exp\{c_5^2\lambda^2\}$ for all $\lambda \in \mathbb{R}$.

   Further the constants $c_1, \ldots, c_5$ differ from each other by an absolute constant.

3. Let $X$ be a mean zero random variable. Show that the following are equivalent.

   (a) $\mathbb{P}\{|X| \geq t\} \leq 2e^{-c_1 t}$, $t \geq 0$.

   (b) $\mathbb{E}\{|X|^p\}^{1/p} \leq c_2 p$ for all $p \geq 1$.

   (c) $\mathbb{E}\{e^{\lambda|X|}\} \leq \exp\{c_3\lambda\}$ for all $|\lambda| \leq c_3^{-1}$.

   (d) $\mathbb{E}\{e^{c_4|X|}\} \leq 2$ for some $c_4$.

   (e) $\mathbb{E}\{e^{\lambda X}\} \leq \exp\{c_5^2\lambda^2\}$ for all $|\lambda| \leq c_5^{-1}$.

   Further the constants $c_1, \ldots, c_5$ differ from each other by an absolute constant.

4. We give an alternative proof of Azuma-Hoeffding inequality.

   (a) Show that for all $x \in [-1, 1]$ and $a > 0$, $e^{ax} \leq cosh(a) + xsinh(a)$ and for all $x$, $cosh(x) \leq e^{x^2/2}$.

   (b) Let $X_1, \ldots, X_n$ be 'mutually uncorrelated' random variables i.e., $\mathbb{E}\{X_{i_1} \ldots X_{i_k}\} = 0$ for all $1 \leq i_1 < \ldots < i_k \leq n$. Assume further tha $|X_i| \leq c_i$ for all $i$. Show that for all $b > 0$,

$$\mathbb{P}\left\{\sum_{i=1}^{n} X_i \geq b\right\} \leq \exp\{-b^2/(2\sum_{i=1}^{n} c_i^2)\}.$$

5. Let $M_i, i = 1, \ldots, n$ be a martingale difference sequence such that $|M_i| \leq c_i$ for all $i$. Show that

$$\mathbb{P}\left\{\sum_{i=1}^{n} M_i \geq b\right\} \leq \exp\{-b^2/(2\sum_{i=1}^{n} c_i^2)\}.$$

6. Let $c_i \in (0, \infty)$ for all $i = 1, \ldots, n$ and $\mathcal{X} = \mathcal{X}_1 \times \ldots \mathcal{X}_n$. Define weighted Hamming distance as

$$d_c(x, y) = \sum_{i=1}^{n} c_i \mathbf{1}[x_i \neq y_i], \; x, y \in \mathcal{X}.$$

Show that $f : \mathcal{X} \to \mathbb{R}$ is Lipschitz w.r.t. $d_c$ (i.e., $|f(x) - f(y)| \leq d_c(x, y)$) iff $\|D_i f\|_\infty \leq c_i$ for all $i$.

7. Let $G(n, p)$ be the Erdös-Rényi random graph with $p = \lambda/n, \lambda \in (0, \infty)$ and $n$ large such that $p \leq 1$. Set $I(n, p)$ to be the number of isolated vertices in $G(n, p)$. Show that as $n \to \infty$,

$$n^{-1} I(n, p) \to e^{-\lambda}, \text{ a.s..}$$

8. Let $\mathcal{X} = [0, 1]^2$ and define the *toroidal metric* on $\mathcal{X}$ as follows : $d(x, y) = \min\{|x - y + z| : z \in \mathbb{Z}^2\}$. Let $X_1, \ldots, X_n$ be i.i.d. uniform points in $\mathcal{X}$. Consider the graph whose vertex set is $1, \ldots, n$ and edge between $i$ and $j$ if $|X_i - X_j| \leq r_n$ where $r_n \in (0, \infty)$ is such that $(n - 1)\pi r_n^2 = \lambda \in (0, \infty)$. Show the following :

   (a) Show that as $n \to \infty$, $deg(i) \xrightarrow{d} Poi(\lambda)$, the Poisson random variables with mean $\lambda$.

   (b) Let $I_n$ be the number of isolated vertices. Show that

$$n^{-1} I_n \to e^{-\lambda}, \text{ a.s..}$$