Stochastic Processes: Assignment 2

Yogeshwaran D.

March 7, 2024

There will be a Quiz from these problems on March 19th (Tuesday) from 9-10 AM and some problems will be discussed in tutorial from 10-11 AM.

1. Prove that the Doob's optional stopping theorem for super-martingales M_t holds under either of these assumptions - The stopping time τ is bounded or M_t is uniformly bounded and τ is a.s. finite.

2. Let $X_1, \ldots, X_n, \ldots$ be centred i.i.d. random variables with finite second moment and τ is a stopping time with finite expectation. Define $S_t = \sum_{i=1}^t X_i$. Show that $\mathbb{E}\{S_\tau^2\} = \mathbb{E}\{\tau\}\,\mathbb{E}\{X_1^2\}$.

3. Consider Polya's urn starting with 1 red ball and 1 green ball. Let G_t be the number of green balls at time t. Show that $\mathbb{P}\{G_t = m+1\} = \frac{(t-m)!m!}{(t+1)!}\binom{t}{m} = \frac{1}{t+1}$.

4. In the problem above, let $M_t = \frac{G_t}{t+2}$. Show that M_t is a Martingale and it converges in distribution to a uniform random variable in $[0,1]$. Does M_t converge a.s. ?

5. Let $Z_n, n \geq 1$ be the branching process with $X_{i,j}$'s having finite mean μ, finite variance σ^2 and $Z_0 = 1$. Let $\mu > 1$. Show that $\liminf_n \mathbb{P}\{\mu^{-n} Z_n > 0\} > 0$.

6. Let A be a $m \times n$ matrix. Show that $|\sum_{i,j} a_{i,j} x_i y_j| \leq 1$ for all $x_i, y_j \in \{+1, -1\}$ iff $|\sum_{i,j} a_{i,j} x_i y_j| \leq 1$ for all $\max_{i,j}\{|x_i|, |y_j|\} \leq 1$.

7. Let A be a symmetric matrix. Assume that for all $x_i \in \{-1, +1\}$, $|\sum_{i,j} a_{i,j} x_i x_j| \leq 1$. Show that for all $N \geq 1$, and $u_i, v_j \in \mathbb{R}^N$ with $\|u_i\| = \|v_j\| = 1$, it holds that
$$|\sum_{i,j} a_{i,j} \langle u_i, v_j \rangle| \leq 2K.$$

8. Let Z be the standard Normal random variable. Show that $\mathbb{E}\{Z^2 \mathbf{1}[|Z| \geq R]\} \leq \sqrt{\frac{2}{\pi}}(R + \frac{1}{R})e^{-R^2/2}$.

9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{H} = \mathbb{L}^2(\mathbb{P}) := \{X : \mathbb{E}\{X^2\} < \infty\}$ with $X \sim Y$ if $X = Y$ a.s.. Show that $\mathbb{H}$ with $\langle X, Y \rangle = \mathbb{E}\{XY\}$ is an inner product space and the norm induces a complete metric.

10. Let P be a stochastic $n \times n$ matrix i.e., $P(i,j) \geq 0$ and $\sum_j P(i,j) = 1$. Show that product of stochastic matrices is a stochastic matrix and convex combination of stochastic matrices is a stochastic matrix i.e., $\sum_i a_i P_i$ is a stochastic matrix if P_i's are stochastic matrices and $a_i \geq 0$ with $\sum_i a_i = 1$.

11. Let $X_t, t \geq 1$ be a Markov chain on V (finite) with reversible (w.r.t. prob. distribution π) transition matrix P. Let X_0 be distributed as π. Show that for all $z_0, \ldots, z_t$,
$$\mathbb{P}\{X_0 = z_0, \ldots, X_t = z_t\} = \mathbb{P}\{X_t = z_0, \ldots, X_0 = z_t\}.$$