Stochastic Processes: Assignment 4

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Submit solutions to Q.4 and Q.5 on Moodle by Friday, 29th March 10 PM.

Q.1,2 and 3 are regarding HMC X_t on a countable set V with stochastic matrix P.

1. For $r \geq 1$, let τ_i^r be the rth return time to the state i and $S_i^r := \tau_i^r - \tau_i^{r-1}$ weith $\tau_i^0 = 0$. Show that $\mathbb{P}\left\{S_i^r = n \mid \tau_i^{r-1} < \infty\right\} = \mathbb{P}\left\{\tau_i^+ = n\right\}$.
2. Let i, j be in the same communicating class. If i is recurrent, then so is j.
3. Show that all states in a finite irreducible Markov chain are positive recurrent.
4. (*Birth-Death Chain:*) Let $V = \mathbb{N} \cup \{0\}$, $p_i, q_i > 0$ and $p_i + q_i + r_i = 1$ for all $i \in V$. X_t be the total population at time t which evolves as follows : If $X_t = i$, then at time $t+1$, there is a birth (i.e., population increases by 1) with probability p_i, death with probability q_i (i.e., population decreases by 1) and nothing happens with probability r_i (i.e., population does not change). Show that X_t is irreducible. Determine the parameters for which X_t has a stationary distribution and determine the stationary distribution. Also determine when is X_t reversible.

   Laplacian: For $f : V \to \mathbb{R}$ and stochastic matrix P, define the Laplacian operator as $\Delta f(x) := \sum_y P(x,y)f(y) - f(x)$.

5. Let X_t be a stochastic process taking values in V, adapted to $\mathcal{F}_t$ and let P be a stochastic matrix. Show that the following are equivalent.
   (a) X_t is a HMC(P).
   (b) For any bounded measurable function $f : V \to \mathbb{R}$, the process $M_t^f := f(X_t) - \sum_{s=0}^{t-1} \Delta f(X_s)$ is a $\mathcal{F}_t$-martingale.
6. A function $h : V \to \mathbb{R}$ is said to be *harmonic* if $\Delta h(x) = 0$ for all $x \in V$. Suppose h is a bounded harmonic function and X_t is HMC(P) then show that $h(X_t)$ is a Martingale.