Stochastic Processes: Assignment 5

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Submit solutions to Q.4, Q.5 and Q.7 on Moodle by Tuesday, 9th March 10 PM.

1. Let P be an irreducible HMC that is either transient or null-recurrent. Show that for all x, y then
$$\lim_{t \to \infty} P^t(x, y) = 0.$$

2. Consider an irreducible positive recurrent HMC with stationary distribution π. Show that for all $x \neq y$,
$$\pi(x)\mathbb{P}_x[\tau_y < \tau_x^+] = \frac{1}{\mathbb{E}_x[\tau_y] + \mathbb{E}_y[\tau_y]}.$$

3. Fix $A \subset V$ and $H_A = \inf\{t \geq 0 : X_t \in A\}$ with $\inf \emptyset = \infty$. Set $h_i := \mathbb{P}_i[H_A < \infty]$. Show that $h_i, i \geq 1$ is the minimal solution to the following set of linear equations:
$$x_i = 1,\, i \in A \;;\; x_i = \sum_j P(i, j)x_j,\, i \notin A.$$

4. In the above question, set $k_i := \mathbb{E}_i[H_A]$. Show that $k_i, i \geq 1$ is the minimal solution to the following set of linear equations:
$$x_i = 0,\, i \in A \;;\; x_i = 1 + \sum_j P(i, j)x_j,\, i \notin A.$$

5. Let X_t be the birth-death chain on $\mathbb{Z}_+$ with $P(x, x+1) = p, P(x, x-1) = 1-p$ and $P(0, 1) = 1$ for $p \in (0, 1)$. Let $h(0) = 1$.
   (a) When $p > 1/2$, show that there is more than one bounded extension of h to $\mathbb{N}$ that is harmonic on $\mathbb{N}$.
   (b) When $p \leq 1/2$, show that there is a unique bounded extension of h to $\mathbb{N}$ that is harmonic on $\mathbb{N}$.

   Let $W \subset V$ be a finite set and $\tau = \tau_{W^c}$ be the hitting time of W^c. Let X_t be irreducible HMC

6. Show that $\tau < \infty$ a.s..

7. If h is a harmonic function on W, then show that $h(X_{t \wedge \tau}), t \geq 0$ is a Martingale.