Stochastic Processes: Assignment 6

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Submit solutions to Q.2, Q.5 and Q.7 on Moodle by Tuesday, 27th April 10 PM.

1. Show that for any exhausting sequence of graphs $G_n \uparrow G$ with sinks Z_n and source $a \in G_n, \forall n \geq 1$, $\lim_{n\to\infty} \mathcal{R}(a \leftrightarrow Z_n)$ exists and is independent of the sequence G_n.

2. Let G be a finite connected network with source a and sink Z. Let $\varepsilon_D(h) := \frac{1}{2}\sum_{x,y} c(x,y)[h(x)-h(y)]^2$ be the *Dirichlet energy* for $h : V \to \mathbb{R}$. Show that

$$\mathcal{C}(a \leftrightarrow Z) := \inf\{\varepsilon_D(h) : h(a) = 1 \; h_Z \equiv 0\},$$

   with the minimizer being the voltage function v from a to Z with $v(a) = 1$.

3. If Π is a cut-set seperating a from Z (possible $Z = \{\infty\}$) in a connected network G, then show that there exists S such that $a \in S$ and $Z \subset S^c$ and $E \cap (S \times S^c) = \Pi$. Furthermore, for a flow θ from a to Z, show that $\|\theta\| = \theta(S, S^c)$.

4. Show that $\mathcal{R}(a \leftrightarrow Z)$ is a concave function of $\{r(e)\}_{e\in E}$ for any finite connected graph G.

5. Show that $\mathcal{R}(x \leftrightarrow y)$, $x, y \in G$ for a finite connected graph G is a metric on V.

6. Let G be a d-regular graph with n vertices and $d > n/2$. Consider the network on G with unit conductances. Let a and z be arbitrary distinct vertices. Show that $\mathcal{R}(a \leftrightarrow z) \leq \frac{2dn}{2d-n}$.

7. In a communication network, packets arrive and are transmitted. At each time step, arrivals and tranmission happen as follows:

   (a) A new packet arrives with probability α or

   (b) An existing packet (if there is one) is transmitted with probability $\beta = 1 - \alpha$. Once the packet is transmitted it exits the network.

   Let X_t be the number of packets in the network at time t. Show that X_t is recurrent iff $\alpha \leq \beta$ i.e., arrival probability is slower than transmission probability. Show that it is positively recurrent if $\alpha < \beta$.